Section 6.1: Graphs and Their Representations

March 17, 2022

Abstract

In this section we are introduced to basic graph definitions and terminology, various kinds of graphs, characteristic features of graphs, and even a few theorems about graphs (for example, we learn when two graphs are the same, or isomorphic, even when they look strikingly different).

We then take a look at planar graphs (in particular at Euler's formula), and computer representations of graphs (adjacency matrices, adjacency lists).

1 Definitions

A graph is defined loosely as a set of nodes, and a set of arcs which connect some of the nodes.

More formally, we have the following

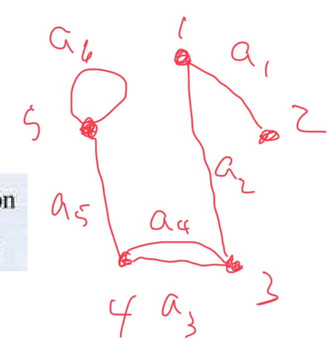
Definition: a **graph** is an ordered triple (N, A, g) where

$N =$	a **nonempty** set of nodes, or vertices
$A =$	a set of arcs, or edges
$g =$	a function associating each arc a with an *unordered* pair {x,y} of nodes, endpoints of the arc.

g is a function $g : A \to \{\{x, y\} | x \in N \text{ and } y \in N\}$.

Example: Practice #1, p. 478.

PRACTICE 1 Sketch a graph having nodes $\{1, 2, 3, 4, 5\}$, arcs $\{a_1, a_2, a_3, a_4, a_5, a_6\}$, and function $g(a_1) = 1–2$, $g(a_2) = 1–3$, $g(a_3) = 3–4$, $g(a_4) = 3–4$, $g(a_5) = 4–5$, and $g(a_6) = 5–5$.

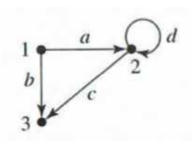


Definition: a **directed graph** is an ordered triple (N, A, g) where

$N =$	a nonempty set of nodes, or vertices
$A =$	a set of arcs, or edges
$g =$	a function associating each arc a with an *ordered* pair (x,y) of nodes.

so g is a function $g : A \to \{(x, y)|x \in N \text{ and } y \in N\}$.

Example: Exercise #1, p. 498. Give the function g that is part of the formal definition of the directed graph shown:



$g(a) = (1,2)$
$g(b) = (1,3)$
$g(c) = (2,3)$
$g(d) = (2,2)$

2 Examples of graphs in action (p. 479-481)

- Road map of Arizona
- Ozone Molecule
- "data flow diagram" for state auto licensing office
- "star topology" for network
- neural network

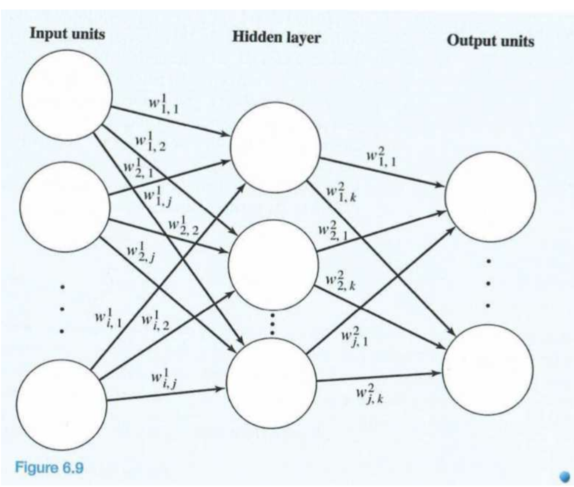
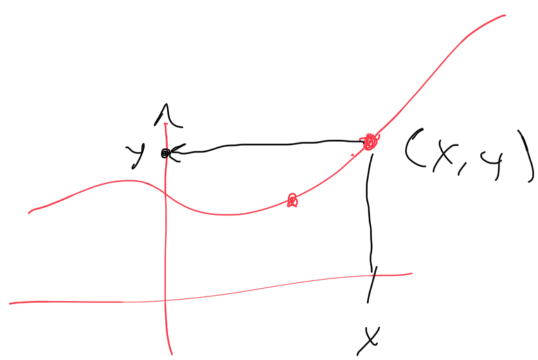


Figure 6.9



- Map of Rabies-infected towns in Connecticut
- Graphs of functions from calculus – Descarte's big idea.

3 Graph Terminology

Take a moment to draw a graph – an object consistent with the above definition(s). We will use a graph terminology handout to classify the graphs. In particular the vocabulary we want to focus on is as follows:

- degree of a vertex – number of incident edges
- adjacent vertices – an edge connects two vertices
- parallel edges – multiple edges connect two vertices

- loop – an edge connects a vertex to itself
- simple – no parallel edges or loops
- complete – simple graph, with every pair of vertices adjacent
- path – sequences of edges from an initial to a terminal vertex
- cycle – a path with initial and terminal vertices the same (no other node repeats)
- reachable – a vertex is reachable from another if a path exists between them.
- connected graph – every vertex is reachable from any other

Example: Exercise #2, p. 498.

Use the graph in the figure to answer the questions that follow.

a. Is the graph simple? Y

b. Is the graph complete? N

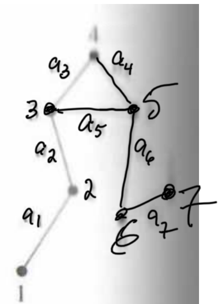
c. Is the graph connected? Y

d. Can you find two paths from 3 to 6? Y

e. Can you find a cycle? 3 - 4 - 5 - 3

f. Can you find an arc whose removal will make the graph acyclic? $\{a_5, a_3, a_4\}$

g. Can you find an arc whose removal will make the graph not connected? $\{a_1, a_2, a_6, a_7\}$



4 Special Graphs

By K_n we will understand the simple, complete graph with n nodes.

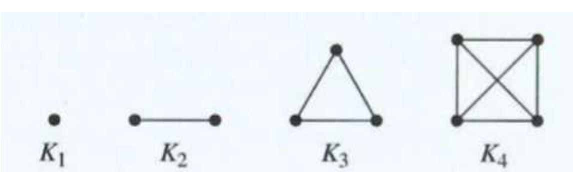
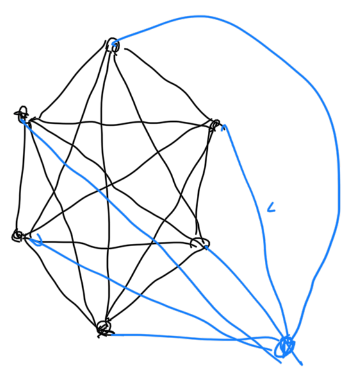


Figure 6.11

$\frac{n(n-1)}{2}$ edges



Example: Exercise #5, p. 498. Draw K_6, emphasizing **symmetry** (a very important and too-little-emphasized concept in mathematics). How many edges does it have? How many edges does K_n have?

A **bipartite complete graph** $K_{m,n}$ is a graph of N nodes which break into two groups, N_1 and N_2, of size m and n respectively, with the property that two nodes x and y are adjacent $\iff$ $x \in N_1$ and $y \in N_2$.

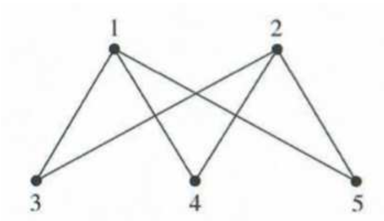
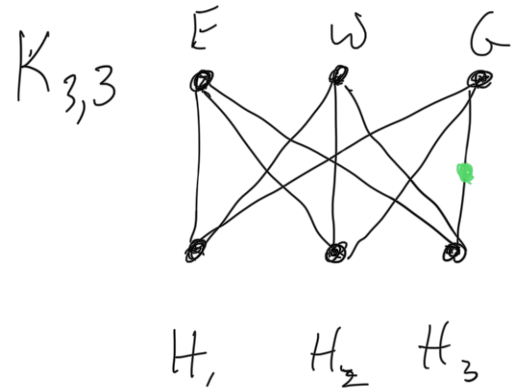


Figure 6.12



Houses & Utilities graph

Example: Exercise #6, p. 498: Draw $K_{3,4}$. How many edges does $K_{m,n}$ have?

m * n edges.

5 Isomorphic Graphs

The idea of isomorphism is that two structures can be "morphed" into each other (they are in some sense identical, up to labelling). Our objective, in general, is to figure out the "morphism" (isomorphism - same form!).

Example: Look at Figure 6.17, p. 485: can you "morph" the two graphs together?

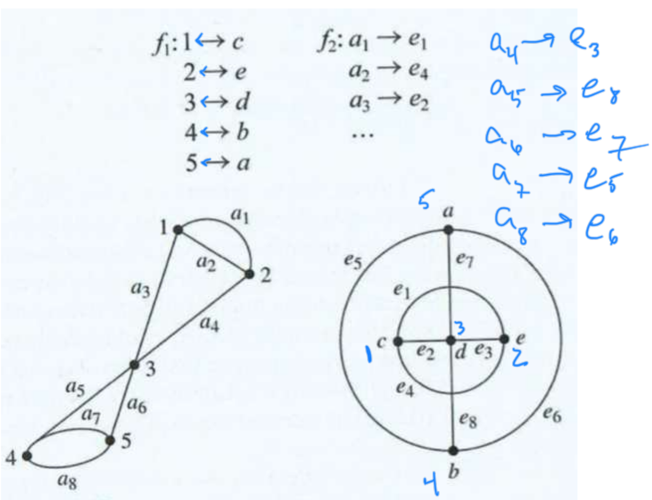


Figure 6.17

Definition: Two graphs (N_1, A_1, g_1) and (N_2, A_2, g_2) are **isomorphic** if there are bijections (one-to-one and onto mappings) $f_1 : N_1 \to N_2$ and $f_2 : A_1 \to A_2$ such that for each arc $a \in A_1$,

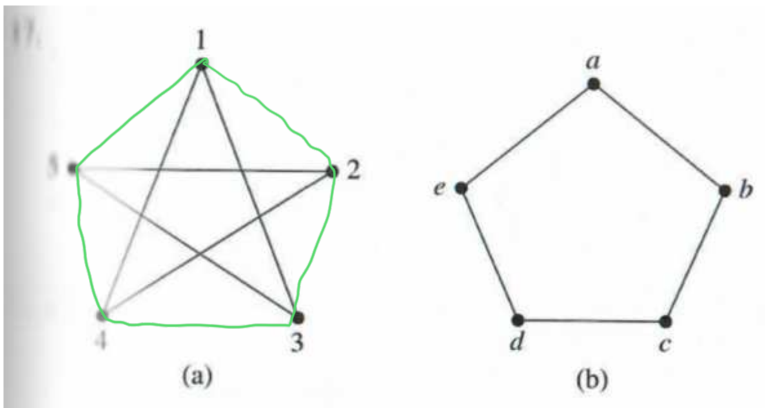
$$\boxed{g_1(a) = \{x, y\} \iff g_2[f_2(a)] = \{f_1(x), f_1(y)\}}$$

(replace braces by parentheses for a directed graph). We can think of the mappings f_1 and f_2 as "relabelling functions". The nodes and arcs are relabelled, preserving all the connectivity of the original graph.

Example: Practice #7, p. 486. If you managed to morph the two graphs in Figure 6.17, then you should be able to "see" the rest of function f_2.

Theorem: Two **simple** graphs (N_1, A_1, g_1) and (N_2, A_2, g_2) are isomorphic if there is a bijection $f : N_1 \to N_2$ such that for any nodes n_i and n_j of N_1, n_i and n_j are adjacent $\iff$ $f(n_i)$ and $f(n_j)$ are adjacent.

Example: Exercise #17, p. 501: the Pentagram.



Here are some tests for determining when two graphs are **not** isomorphic:

(I) The graphs don't have the same number of nodes.

(II) The graphs don't have the same number of arcs.

(III) One graph is connected and the other isn't.

(IV) One graph has a node of degree k and the other doesn't.

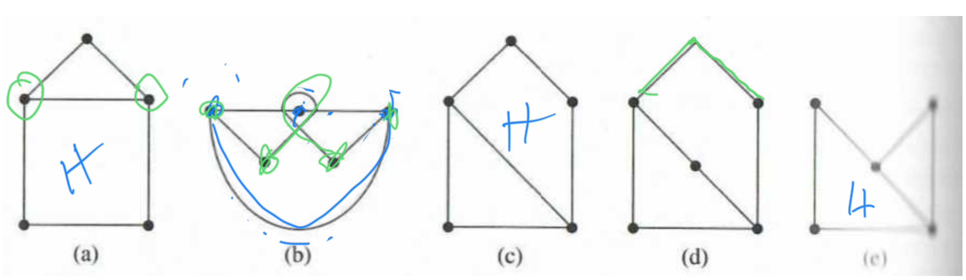
(V) One graph has parallel arcs and the other doesn't.

(VI) One graph has loops and the other doesn't.

(VII) One graph has cycles and the other doesn't.

This list is not complete, however: sometimes things get trickier than this (as shown in Example 12, p. 487).

Example: Exercise #14, p. 500.



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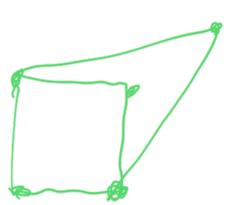
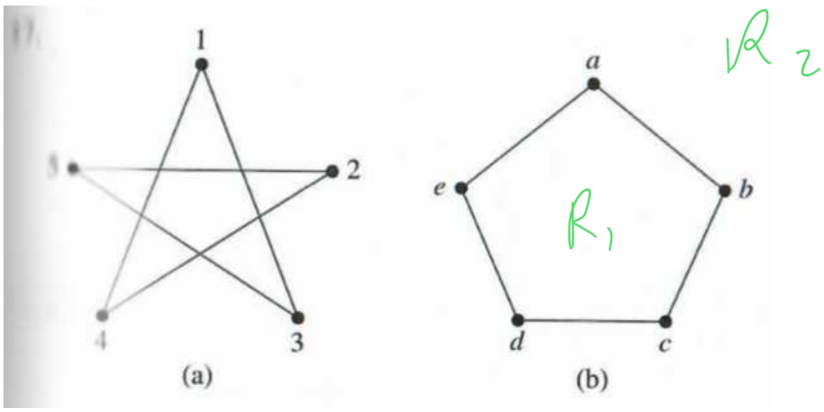


6 Planar Graphs

A **planar graph** is one which can be drawn in two-dimensions so that its arcs intersect only in nodes. “Designers of integrated circuits want all components in one layer of a chip to form a planar graph so that no connections cross.” (p. 487)

Example: Revisit #17, p. 501.



$r = 2$

$a = 5$

$n = 5$

Euler’s Formula for simple, connected planar graphs states that

$$r - a + n = 2$$

where n is the number of nodes, a is the number of arcs, and r is the number of regions (including the infinite region surrounding the graph).

Think “ran to” to remember the formula....

Check out the author’s proof of the theorem (p. 488-489): Hey! What’s induction doing in here? Euler’s formula is proven by induction, on a, the number of arcs, and a consideration of cases (node of degree 1; no node of degree 1). Figure 6.22 illustrates the base case and the two cases essential to the proof of the inductive step.

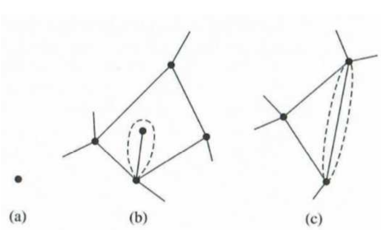


Figure 6.22

Note: Leonhard Euler. Born: 15 April, 1707 in Basel, Switzerland; died 18 Sept, 1783 in St Petersburg, Russia. He was so prolific that his work is still being compiled. He went blind in his old age, and became even more prolific! He was an incredible calculating machine.

Example: Revisit #17, p. 501, for a check.

The following theorem provides some estimates on the relationship between the number of arcs and nodes that a planar graph may possess:

$r - a + n = 2$

Theorem: For a simple, connected, planar graph with n nodes and a arcs,

$r = a - n + 2$

$\leq \frac{2}{3}a$

(I) If the planar representation divides the plane into r regions, then

$$r - a + n = 2$$

(II) If $n \geq 3$, then

$$\boxed{a \leq 3(n-2)}$$

This is a consequence of Euler's formula and the inequality $2a \geq 3r$. This is a statement about **region edges**. The best any arc can do is contribute to two different region edges (so at **most** $2a$ total); and the **least** number of region edges that a region can have is 3 ($3r$ total). Hence, the most number of region edges possible is $2a$, and the least is $3r$. This leads to our inequality.

$2a \geq \boxed{3r}$

$r \leq \frac{2}{3}a$

(III) If $n \geq 3$ and there are no cycles of length 3, then

$$\boxed{a \leq 2(n-2)}$$

$2a \geq 4r$

This is a consequence of Euler's formula and the inequality $2a \geq 4r$, because each region now requires **four** arcs (at least) to define itself.

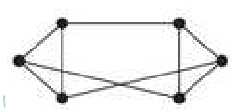
From part (II) of this theorem we can deduce that K_5 is not planar, since it has 5 nodes, and 10 arcs, and $10 \not\leq 3(5-2) = 9$.

From part (III) of this theorem we can deduce that $K_{3,3}$ is not planar, since it has 6 nodes, and 9 arcs, and no cycles of length 3: $9 \not\leq 2(6-2) = 8$.

Interestingly enough, it has been shown that any graph failing to be planar has a copy of either K_5 or $K_{3,3}$ in it (Kuratowski's Theorem, p. 491).

Example: Exercise #28, p. 502.

Prove that the following graph is a planar graph.



7 Computer Representations of Graphs

We want to examine two different representations of graphs by a computer:

- the adjacency matrix, and
- the adjacency list.

A **matrix** is basically a spreadsheet: a rectangular data set of numbers indexed by rows and columns.

An **adjacency matrix** for a graph with N nodes is of size N by N, where the rows and columns of the matrix represent the vertices. If the graph is undirected, then the element a_{ij} of the matrix is non-zero $\iff$ nodes i and j are adjacent; if directed, then the element a_{ij} of the matrix is non-zero $\iff$ there is an arc **from** node i **to** node j.

In our textbook, the element of the matrix $a_{ij} = p$, the number of arcs meeting the criteria above.

Example: Practice #16, p. 493. Complete the adjacency matrix for Figure 6.25:

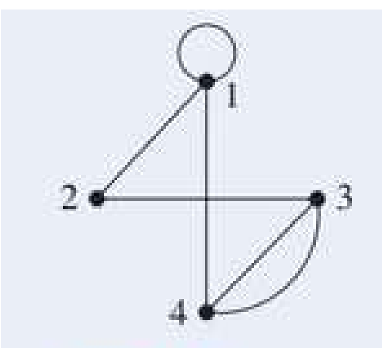


Figure 6.25

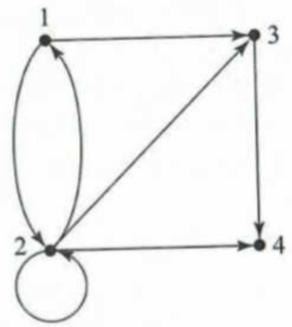
$$
\begin{array}{c|cccc}
 & 1 & 2 & 3 & 4 \\
\hline
1 & 1 & 1 & 0 & 1 \\
2 & 1 & 0 & 1 & 0 \\
3 & 0 & 1 & 0 & 2 \\
4 & 1 & 0 & 2 & 0
\end{array}
$$

For an undirected graph the adjacency matrix is **symmetric** (which means that we can reduce storage by about half); for a directed graph, the matrix may well be unsymmetric.

Let's look at a nice web page, with an example of a directed graph. Notice that the node of departure is the row node (in this webpage, and according to our author's convention).

Example: Exercise #42, p. 504. Write the adjacency matrix for the given graph:

42.



	1	2	3	4
1	0	1	1	0
2	1	1	1	1
3	0	0	0	1
4	0	0	0	0

- The 1990 commuting patterns page

  might be modelled as a directed, weighted graph. Its adjacency matrix would be exactly the numerical portion of this table, and it would be a *full* matrix.

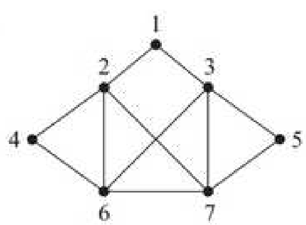
- A map of Rabies-infected towns in Connecticut

  gives rise to an undirected graph. The towns are nodes, and an arc is created if two towns are adjacent. This will lead to a *sparse* symmetric adjacency matrix, however, as very few towns are adjacent to any particular town.

An **adjacency list** might be a better storage method for graphs with relatively few arcs: we effectively store only the non-zero entries of the adjacency matrix, in a linked list:

Example: Exercise #55, p. 505. Draw the adjacency list for the graph of Ex. 39:

39.



The redundancy in drawing the adjacency list for an undirected graph is evident. This is eliminated for a directed graph:

Example: Exercise #64, p. 505.