Problem 4: We measure the mass of a radioactive element over a period of years:

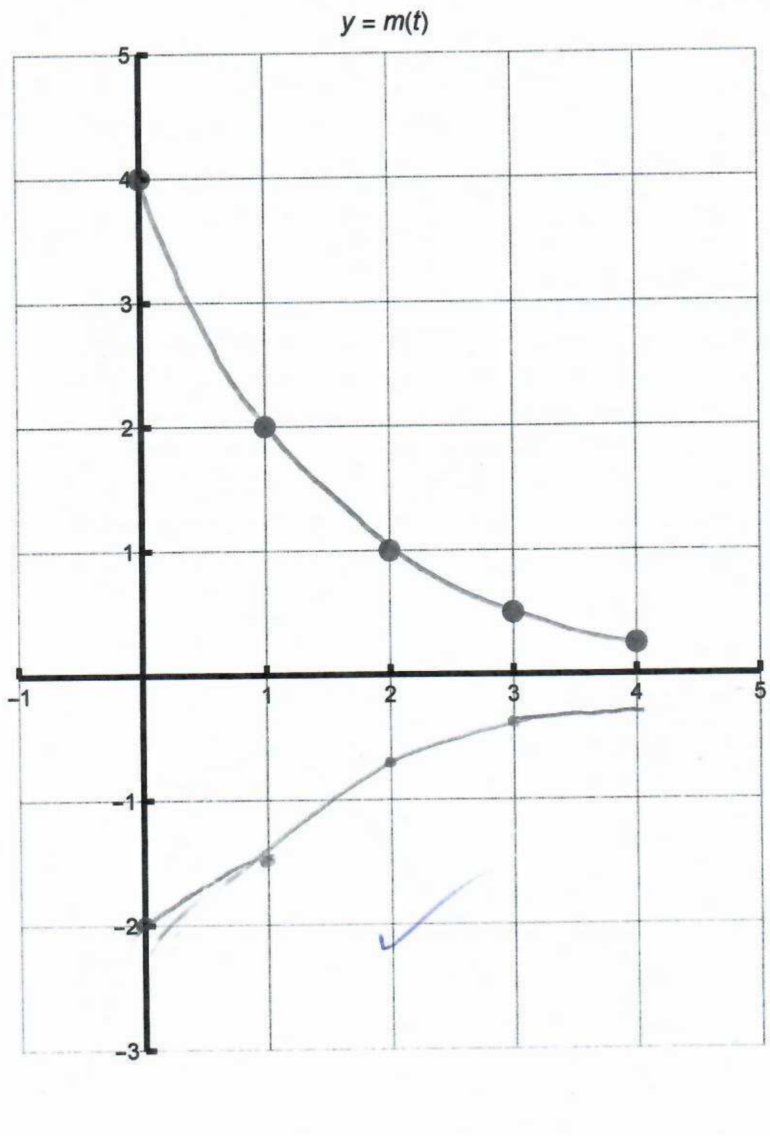
time t (years)	0	1	2	3	4
height $m(t)$ (kg)	4	2	1	0.5	0.25
$\approx m'(t)$	-2	-1.5	$-.75$	$-.375$	$-.25$
$\approx m''(t)$	$.5$	$.625$	$.5625$	$.25$	$.125$
method	forward	centered	centered	centered	backward

a. (10 pts) Choose the best method to approximate the first and second derivatives of m at each point, and so fill in the table above.

b. (2 pts) What is the average rate of change over the entire interval, and what are its units?

Average rate of change $= \dfrac{0.25-4}{4-0} = \boxed{-.9375 \text{ kg/year}}$

c. (8 pts) Graph the approximate derivative function below, using your estimates for the derivative. (Does it make sense? Do you have any qualms? Why or why not?)



It makes sense for the most part. the first point (0,-2) throws the graph off a little bit. Since it isn't centered, which is more accurate than forwards, the point isn't as exact which is why the graph looks slightly off on [0,1]. It is safe to assume the same for (4,-.25). Although it isn't as noticeable, the graph is again, a little off on [3,4] because backwards method was used instead of centered.

20

Problem 5: **Explain with reasons** in each case a-c; figures are encouraged.

a. (2 pts) If a function is continuous at a point, is it differentiable there?

this is true for ~~every single function~~ except $f(x) = |x|$. Since this function has a continuous point that doesn't have a derivative because that point is at a corner ∨, which can't / doesn't have an instantaneous velocity

b. (2 pts) If a function is differentiable at a point, is it continuous there?

This is true. For a point to be differentiable, it must be continuous. ∴ If a function is differentiable at a given point, the function must also be continuous at that point.

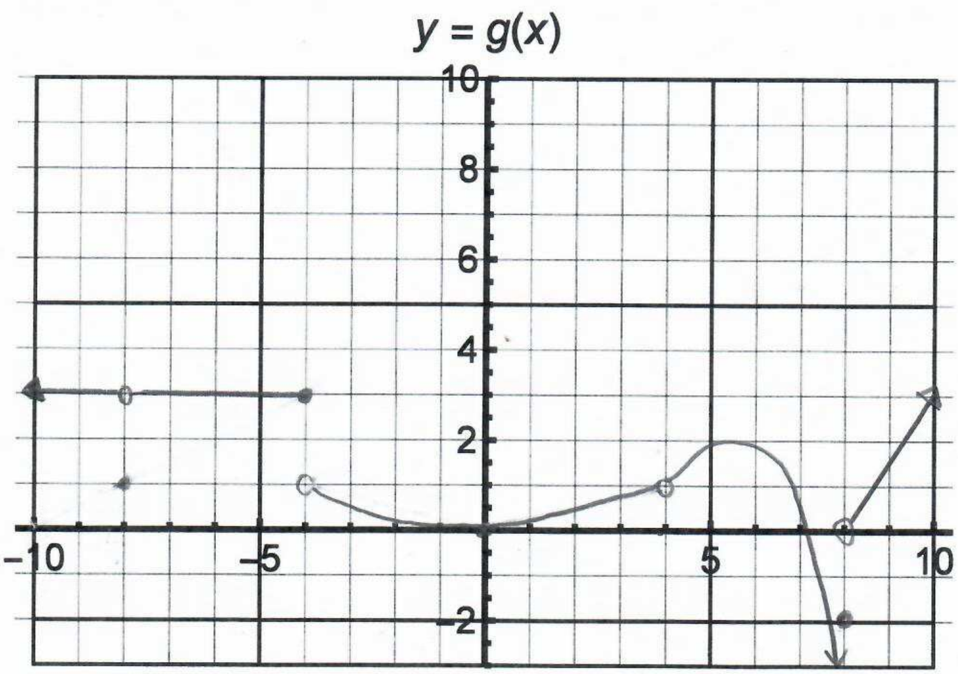
c. (6 pts) What problems can prevent a function from having a derivative at a point (the more the merrier)?

Any time a function is discontinuous, it doesn't have at the discontinuity; whether it is a removable, jump, or infinite discontinuity. It also can't have a derivative at the point of the corner of the absolute value function. It should work for all other continuous function and continuous points.

d. (10 pts) Draw the graph of a function g consistent with the following:

$g(-8) = 1$
$g(-4) = 3$
$g(0) = 0$
$g(4) = DNE$
$g(8) = -2$

$\lim_{x\to-8} g(x) = 3$
$\lim_{x\to-4^-} g(x) = 3$
$\lim_{x\to-4^+} g(x) = 1$
$\lim_{x\to0} g(x) = 0$
$\lim_{x\to4} g(x) = 1$
$\lim_{x\to8^-} g(x) = DNE$
$\lim_{x\to8^+} g(x) = 0$



20

roblem 5: **Explain with reasons** in each case a-c; figures are encouraged.

a. (2 pts) If a function is continuous at a point, is it differentiable there?

No, as it may possibly be a corner in a function.

b. (2 pts) If a function is differentiable at a point, is it continuous there?

Yes, since the derivative can be obtained at that point, it must be continuous.

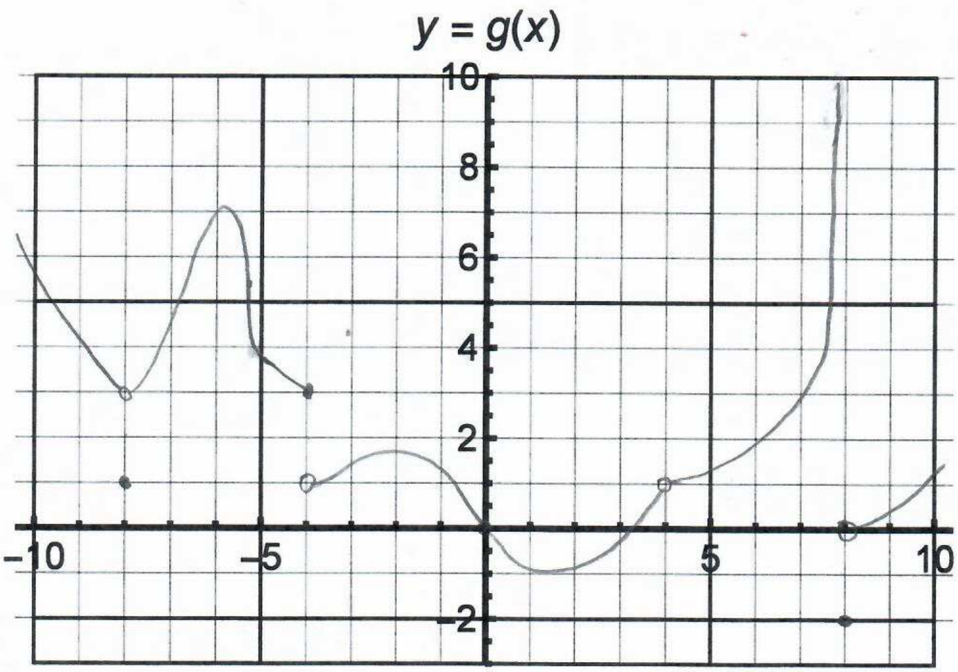
c. (6 pts) What problems can prevent a function from having a derivative at a point (the more the merrier)?

Corners in graphs
holes
asymptotes
Being undefined at a point.

d. (10 pts) Draw the graph of a function g consistent with the following:

$g(-8) = 1$
$g(-4) = 3$
$g(0) = 0$
$g(4) = DNE$
$g(8) = -2$

$\lim_{x\to -8} g(x) = 3$
$\lim_{x\to -4^-} g(x) = 3$
$\lim_{x\to -4^+} g(x) = 1$
$\lim_{x\to 0} g(x) = 0$
$\lim_{x\to 4} g(x) = 1$
$\lim_{x\to 8^-} g(x) = DNE$
$\lim_{x\to 8^+} g(x) = 0$



20

Problem 6: Assume that $F(x) = f(x) + g(x)$, where f and g are differentiable functions of x.

a. (10 pts) Using the limit definition, show that

$$F'(x) = f'(x) + g'(x)$$

$$f'(x) = \lim_{h \to 0} \frac{(f(x+h) + g(x+h)) - (f(x) + g(x))}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h) - f(x) + g(x+h) - g(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

$$= f'(x) + g'(x)$$

✓

b. (10 pts) Show how to apply the sum rule (and other rules) to compute the derivative of

$$F(x) = 3e^x + x^2 - \sin(x) + x\cos(x)$$

without using the limit definition. You must show your work!

$\frac{d}{dx}(f(x) \cdot g(x)) = f'(x)g(x) + g'(x)f(x)$ $\quad \therefore 3 * e^x$

$= 0 \cdot e^x + e^x \cdot 3$

$= 3e^x$ ✓

$1 \cdot \cos(x) + -\sin(x) \cdot x$

$\cos(x) - \sin(x) \cdot x$

$\frac{d}{dx} x^k = k x^{k-1}$ $\quad \therefore x^2$

$= 2x$ ✓

$\frac{d}{dx} \sin(x) = \cos(x)$ $\quad \therefore -\sin(x) = -\cos(x)$ ✓

$\frac{d}{dx} \cos(x) = -\sin(x)$

good work!

$\therefore f'(x) = 3e^x + 2x - \cos(x) + \cancel{\cos(x)} - \sin(x) \cdot x$

$= 3e^x + 2x - x\sin(x)$ ✓

20

Problem 7: Consider the function $f(x) = x^{\frac{1}{3}}$.

- (8 pts) Find the equation of the tangent line to the graph of f at $x = 27$. Of course you should write it in point-slope form.

$f(x) = x^{1/3}$

$f'(x) = 1/3\, x^{-2/3}$

$f'(27) = \frac{1}{3}(27)^{-2/3} \rightarrow \frac{1}{27} = m$

$f(27) = 27^{1/3} = 3$

PS Form

$y - y_1 = m(x - x_1)$

$(27, 3)$

$\Rightarrow y - 3 = \frac{1}{27}(x - 27)$

good

- (8 pts) Use the local linearization to give the exact approximation to $(27.03)^{\frac{1}{3}}$. How well does it compare to this 17-digit approximation, 3.0011106998423154?

LL: $y = f'(a)(x-a) + f(a)$

$\frac{1}{27}(27.03 - 27) + 3 \doteq 3.00\overline{1111111}$

Yes!

oblem 7: Consider the function $f(x) = x^{\frac{1}{3}}$.

- (8 pts) Find the equation of the tangent line to the graph of f at $x = 27$. Of course you should write it in point-slope form.

$f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$

$f'(27) = .037 = \frac{1}{27}$

$f(27) = 3$

$y - 3 = .037(x - 27)$

- (8 pts) Use the local linearization to give the exact approximation to $(27.03)^{\frac{1}{3}}$. How well does it compare to this 17-digit approximation, 3.0011106998423154?

$y - 3 = .037(27.03 - 27)$

$y = 3.00111$

It has a precision of 5.

This is a consequence of you rounding $f'(27)$.

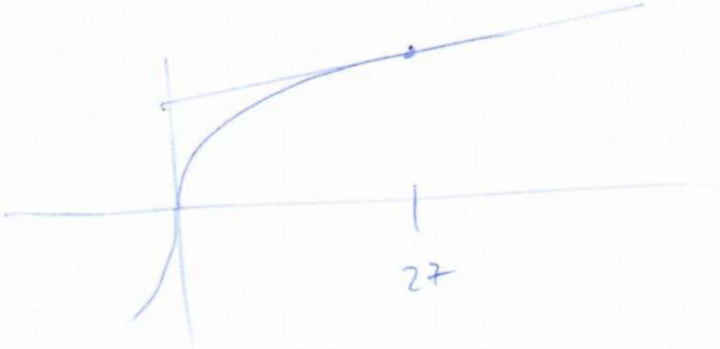
- (4 pts) Is your answer an over- or an under-estimate? Justify your answer.

It's an underestimation because it terminates before the above approximation without any sort of rounding.

-2

It's an over-estimate because of the concavity!

- (4 pts) Is your answer an over- or an under-estimate? Justify your answer.



It's an over-estimate, because f is concave down at $x = 27$ (so the linearization is above).