For reasons that will be clear momentarily, we'll also establish two additional cases, $P(9)$ and $P(10)$, by the equations

$$9 = 3 + 3 + 3$$
$$10 = 5 + 5$$

Now we assume that $P(r)$ is true, that is, r can be written as a sum of 3s and 5s, for any r, $8 \leq r \leq k$, and consider $P(k + 1)$. We may assume that $k + 1$ is at least 11, because we have already proved $P(r)$ true for $r = 8$, 9, and 10. If $k + 1 \geq 11$, then $(k + 1) - 3 = k - 2 \geq 8$. Thus $k - 2$ is a legitimate r value, and by the inductive hypothesis, $P(k - 2)$ is true. Therefore $k - 2$ can be written as a sum of 3s and 5s, and adding an additional 3 gives us $k + 1$ as a sum of 3s and 5s. This verifies that $P(k + 1)$ is true and completes the proof. ●

PRACTICE 9

a. Why are the additional cases $P(9)$ and $P(10)$ proved separately in Example 24?
b. Why can't the first principle of induction be used in the proof of Example 24? ■

REMINDER
Use the second principle of induction when the $k + 1$ case depends on results farther back than k.

As a general rule, the first principle of mathematical induction applies when information about "one position back" is enough, that is, when the truth of $P(k)$ is enough to prove the truth of $P(k + 1)$. The second principle applies when information about "one position back" isn't good enough; that is, you can't prove that $P(k + 1)$ is true just because you know $P(k)$ is true, but you can prove $P(k + 1)$ true if you know that $P(r)$ is true for one or more values of r that are "farther back" than k.

SECTION 2.2 REVIEW

TECHNIQUES

- Use the first principle of induction in proofs.
- Use the second principle of induction in proofs.

MAIN IDEAS

- Mathematical induction is a technique to prove properties of positive integers.
- An inductive proof need not begin with 1.
- Induction can be used to prove statements about quantities whose values are arbitrary nonnegative integers.
- The first and second principles of induction each prove the same conclusion, but one approach may be easier to use than the other in a given situation.

EXERCISES 2.2

1. For all positive integers, let $P(n)$ be the equation

$$2 + 6 + 10 + \cdots + (4n - 2) = 2n^2$$

a. Write the equation for the base case $P(1)$ and verify that it is true.
b. Write the inductive hypothesis $P(k)$.
c. Write the equation for $P(k + 1)$.
d. Prove that $P(k + 1)$ is true.

2. For all positive integers, let $P(n)$ be the equation

$$2 + 4 + 6 + \cdots + 2n = n(n + 1)$$

a. Write the equation for the base case $P(1)$ and verify that it is true.
b. Write the inductive hypothesis $P(k)$.
c. Write the equation for $P(k + 1)$.
d. Prove that $P(k + 1)$ is true.

In Exercises 3–26, use mathematical induction to prove that the statements are true for every positive integer n. [*Hint*: In the algebra part of the proof, if the final expression you want has factors and you can pull those factors out early, do that instead of multiplying everything out and getting some humongous expression.]

3. $1 + 5 + 9 + \cdots + (4n - 3) = n(2n - 1)$
4. $1 + 3 + 6 + \cdots + \dfrac{n(n+1)}{2} = \dfrac{n(n+1)(n+2)}{6}$
5. $4 + 10 + 16 + \cdots + (6n - 2) = n(3n + 1)$
6. $5 + 10 + 15 + \cdots + 5n = \dfrac{5n(n+1)}{2}$
7. $1^2 + 2^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$
8. $1^3 + 2^3 + \cdots + n^3 = \dfrac{n^2(n+1)^2}{4}$
9. $1^2 + 3^2 + \cdots + (2n - 1)^2 = \dfrac{n(2n-1)(2n+1)}{3}$
10. $1^4 + 2^4 + \cdots + n^4 = \dfrac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$
11. $1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \cdots + n(n + 2) = \dfrac{n(n+1)(2n+7)}{6}$
12. $1 + a + a^2 + \cdots + a^{n-1} = \dfrac{a^n - 1}{a - 1}$ for $a \neq 0, a \neq 1$
13. $\dfrac{1}{1 \cdot 2} + \dfrac{1}{2 \cdot 3} + \dfrac{1}{3 \cdot 4} + \cdots + \dfrac{1}{n(n+1)} = \dfrac{n}{n+1}$
14. $\dfrac{1}{1 \cdot 3} + \dfrac{1}{3 \cdot 5} + \dfrac{1}{5 \cdot 7} + \cdots + \dfrac{1}{(2n-1)(2n+1)} = \dfrac{n}{2n+1}$
15. $1^2 - 2^2 + 3^2 - 4^2 + \cdots + (-1)^{n+1}n^2 = \dfrac{(-1)^{n+1}(n)(n+1)}{2}$
16. $2 + 6 + 18 + \cdots + 2 \cdot 3^{n-1} = 3^n - 1$
17. $2^2 + 4^2 + \cdots + (2n)^2 = \dfrac{2n(n+1)(2n+1)}{3}$
18. $1 \cdot 2^1 + 2 \cdot 2^2 + 3 \cdot 2^3 + \cdots + n \cdot 2^n = (n - 1)2^{n+1} + 2$
19. $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n + 1) = \dfrac{n(n+1)(n+2)}{3}$

20. $1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \cdots + n(n + 1)(n + 2) = \dfrac{n(n + 1)(n + 2)(n + 3)}{4}$

21. $\dfrac{1}{1 \cdot 4} + \dfrac{1}{4 \cdot 7} + \dfrac{1}{7 \cdot 10} + \cdots + \dfrac{1}{(3n - 2)(3n + 1)} = \dfrac{n}{3n + 1}$

22. $1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \cdots + n \cdot n! = (n + 1)! - 1$ where $n!$ is the product of the positive integers from 1 to n.

23. $1 + 4 + 4^2 + \cdots + 4^n = \dfrac{4^{n+1} - 1}{3}$

24. $1 + x + x^2 + \cdots + x^n = \dfrac{x^{n+1} - 1}{x - 1}$ where x is any integer > 1

25. $1 + 4 + 7 + 10 + \cdots + (3n - 2) = \dfrac{n(3n - 1)}{2}$

26. $1 + [x \cdot 2 - (x - 1)] + [x \cdot 3 - (x - 1)] + \cdots + [x \cdot n - (x - 1)] = \dfrac{n[xn - (x - 2)]}{2}$ where x is any integer ≥ 1

27. A *geometric progression* (*geometric sequence*) is a sequence of terms where there is an initial term a and each succeeding term is obtained by multiplying the previous term by a *common ratio* r. Prove the formula for the sum of the first n ($n \geq 1$) terms of a geometric sequence where $r \neq 1$:

$$a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a - ar^n}{1 - r}$$

28. An *arithmetic progression* (*arithmetic sequence*) is a sequence of terms where there is an initial term a and each succeeding term is obtained by adding a *common difference* d to the previous term. Prove the formula for the sum of the first n ($n \geq 1$) terms of an arithmetic sequence:

$$a + (a + d) + (a + 2d) + \cdots + [a + (n - 1)d] = \frac{n}{2}[2a + (n - 1)d]$$

29. Using Exercises 27 and 28, find an expression for the value of the following sums.
   a. $2 + 2 \cdot 5 + 2 \cdot 5^2 + \cdots + 2 \cdot 5^9$
   b. $4 \cdot 7 + 4 \cdot 7^2 + 4 \cdot 7^3 + \cdots + 4 \cdot 7^{12}$
   c. $1 + 7 + 13 + \cdots + 49$
   d. $12 + 17 + 22 + 27 + \cdots + 92$

30. Prove that

$$(-2)^0 + (-2)^1 + (-2)^2 + \cdots + (-2)^n = \frac{1 - 2^{n+1}}{3}$$

for every positive odd integer n.

31. Prove that $n^2 > n + 1$ for $n \geq 2$.
32. Prove that $n^2 \geq 2n + 3$ for $n \geq 3$.
33. Prove that $n^2 > 5n + 10$ for $n > 6$.
34. Prove that $2^n > n^2$ for $n \geq 5$.

In Exercises 35–40, $n!$ is the product of the positive integers from 1 to n.

35. Prove that $n! > n^2$ for $n \geq 4$.

36. Prove that $n! > n^3$ for $n \geq 6$.
37. Prove that $n! > 2^n$ for $n \geq 4$.
38. Prove that $n! > 3^n$ for $n \geq 7$.
39. Prove that $n! \geq 2^{n-1}$ for $n \geq 1$.
40. Prove that $n! < n^n$ for $n \geq 2$.
41. Prove that $(1 + x)^n > 1 + x^n$ for $n > 1, x > 0$.
42. Prove that $\left(\frac{a}{b}\right)^{n+1} < \left(\frac{a}{b}\right)^n$ for $n \geq 1$ and $0 < a < b$.
43. Prove that $1 + 2 + \cdots + n < n^2$ for $n > 1$.
44. Prove that $1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} < 2 - \frac{1}{n}$ for $n \geq 2$
45. a. Try to use induction to prove that

$$1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} < 2 \qquad \text{for } n \geq 1$$

What goes wrong?

b. Prove that

$$1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} = 2 - \frac{1}{2^n} \qquad \text{for } n \geq 1$$

thus showing that

$$1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} < 2 \qquad \text{for } n \geq 1$$

46. Prove that

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^n} \geq 1 + \frac{n}{2} \qquad \text{for } n \geq 1$$

(Note that the denominators increase by 1, not by powers of 2.)

For Exercises 47–58, prove that the statements are true for every positive integer.

47. $2^{3n} - 1$ is divisible by 7.
48. $3^{2n} + 7$ is divisible by 8.
49. $7^n - 2^n$ is divisible by 5.
50. $13^n - 6^n$ is divisible by 7.
51. $2^n + (-1)^{n+1}$ is divisible by 3.
52. $2^{5n+1} + 5^{n+2}$ is divisible by 27.
53. $3^{4n+2} + 5^{2n+1}$ is divisible by 14.
54. $7^{2n} + 16n - 1$ is divisible by 64.
55. $10^n + 3 \cdot 4^{n+2} + 5$ is divisible by 9.
56. $n^3 - n$ is divisible by 3.
57. $n^3 + 2n$ is divisible by 3.
58. $x^n - 1$ is divisible by $x - 1$ for $x \neq 1$.

59. Prove DeMoivre's Theorem:

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

for all $n \geq 1$. *Hint:* Recall the addition formulas from trigonometry:

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

60. Prove that

$$\sin\theta + \sin 3\theta + \cdots + \sin(2n-1)\theta = \frac{\sin^2 n\theta}{\sin\theta}$$

for all $n \geq 1$ and all θ for which $\sin\theta \neq 0$.

61. Use induction to prove that the product of any three consecutive positive integers is divisible by 3.

62. Suppose that exponentiation is defined by the equation

$$x^j \cdot x = x^{j+1}$$

for any $j \geq 1$. Use induction to prove that $x^n \cdot x^m = x^{n+m}$ for $n \geq 1$, $m \geq 1$. (*Hint:* Do induction on m for a fixed, arbitrary value of n.)

63. According to Example 20, it is possible to use angle irons to tile a 4×4 checkerboard with the upper right corner removed.Sketch such a tiling.

64. Example 20 does not cover the case of checkerboards that are not sized by powers of 2. Determine whether it is possible to tile a 3×3 checkerboard.

65. Prove that it is possible to use angle irons to tile a 5×5 checkerboard with the upper left corner removed.

66. Find a configuration for a 5×5 checkerboard with one square removed that is not possible to tile; explain why this is not possible.

67. Consider n infinitely long straight lines, none of which are parallel and no three of which have a common point of intersection. Show that for $n \geq 1$, the lines divide the plane into $(n^2 + n + 2)/2$ separate regions.

68. A string of 0*s* and 1*s* is to be processed and converted to an even-parity string by adding a parity bit to the end of the string.(For an explanation of the use of parity bits, see Example 30 in Chapter 9.) The parity bit is initially 0. When a 0 character is processed, the parity bit remains unchanged. When a 1 character is processed, the parity bit is switched from 0 to 1 or from 1 to 0. Prove that the number of 1*s* in the final string, that is, including the parity bit, is always even. (*Hint:* Consider various cases.)

69. What is wrong with the following "proof" by mathematical induction? We will prove that for any positive integer n, n is equal to 1 more than n. Assume that $P(k)$ is true.

$$k = k + 1$$

Adding 1 to both sides of this equation, we get

$$k + 1 = k + 2$$

Thus,

$$P(k + 1) \text{ is true}$$

70. What is wrong with the following "proof" by mathematical induction?

> We will prove that all computers are built by the same manufacturer. In particular, we will prove that in any collection of n computers where n is a positive integer, all the computers are built by the same manufacturer. We first prove $P(1)$, a trivial process, because in any collection consisting of

one computer, there is only one manufacturer. Now we assume $P(k)$; that is, in any collection of k computers, all the computers were built by the same manufacturer. To prove $P(k + 1)$, we consider any collection of $k + 1$ computers. Pull one of these $k + 1$ computers (call it HAL) out of the collection. By our assumption, the remaining k computers all have the same manufacturer. Let HAL change places with one of these k computers. In the new group of k computers, all have the same manufacturer. Thus, HAL's manufacturer is the same one that produced all the other computers, and all $k + 1$ computers have the same manufacturer.

71. An obscure tribe has only three words in its language, *moon*, *noon*, and *soon*. New words are composed by juxtaposing these words in any order, as in *soonnoonmoonnoon*. Any such juxtaposition is a legal word.
    a. Use the first principle of induction (on the number of subwords in the word) to prove that any word in this language has an even number of o's.
    b. Use the second principle of induction (on the number of subwords in the word) to prove that any word in this language has an even number of o's.
72. A *simple closed polygon* consists of n points in the plane joined in pairs by n line segments; each point is the endpoint of exactly 2 line segments. Following are two examples.

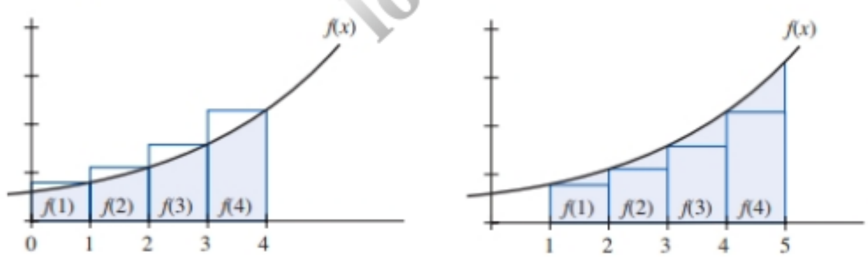
    a. Use the first principle of induction to prove that the sum of the interior angles of an n-sided simple closed polygon is $(n - 2)180°$ for all $n \geq 3$.
    b. Use the second principle of induction to prove that the sum of the interior angles of an n-sided simple closed polygon is $(n - 2)180°$ for all $n \geq 3$.
73. The Computer Science club is sponsoring a jigsaw puzzle contest. Jigsaw puzzles are assembled by fitting 2 pieces together to form a small block, adding a single piece to a block to form a bigger block, or fitting 2 blocks together. Each of these moves is considered a step in the solution. Use the second principle of induction to prove that the number of steps required to assemble an n-piece jigsaw puzzle is $n - 1$.
74. OurWay Pizza makes only two kinds of pizza, pepperoni and vegetarian. Any pizza of either kind comes with an even number of breadsticks (not necessarily the same even number for both kinds). Any order of 2 or more pizzas must include at least 1 of each kind. When the delivery driver goes to deliver an order, he or she puts the completed order together by combining 2 suborders—picking up all the pepperoni pizzas from 1 window and all the vegetarian pizzas from another window. Prove that for a delivery of n pizzas, $n \geq 1$, there are an even number of breadsticks included.
75. Consider propositional wffs that contain only the connectives $\wedge$, $\vee$, and $\rightarrow$ (no negation) and where wffs must be parenthesized when joined by a logical connective. Count each statement letter, connective, or parenthesis as one symbol. For example, $((A) \wedge (B)) \vee ((C) \wedge (D))$ is such a wff, with 19 symbols. Prove that any such wff has an odd number of symbols.
76. In any group of k people, $k \geq 1$, each person is to shake hands with every other person. Find a formula for the number of handshakes, and prove the formula using induction.

77. Prove that any amount of postage greater than or equal to 2 cents can be built using only 2-cent and 3-cent stamps.
78. Prove that any amount of postage greater than or equal to 12 cents can be built using only 4-cent and 5-cent stamps.
79. Prove that any amount of postage greater than or equal to 14 cents can be built using only 3-cent and 8-cent stamps.
80. Prove that any amount of postage greater than or equal to 42 cents can be built using only 4-cent and 15-cent stamps.
81. Prove that any amount of postage greater than or equal to 64 cents can be built using only 5-cent and 17-cent stamps.
82. Your bank ATM delivers cash using only \$20 and \$50 bills. Prove that you can collect, in addition to \$20, any multiple of \$10 that is \$40 or greater.

Exercises 83–84 require familiarity with ideas from calculus. Exercises 1–26 give exact formulas for the sum of terms in a sequence that can be expressed as $\sum_{m=1}^{n} f(m)$. Sometimes it is difficult to find an exact expression for this summation, but if the value of $f(m)$ increases monotonically, integration can be used to find upper and lower bounds on the value of the summation. Specifically,

$$\int_0^n f(x)dx \leq \sum_{m=1}^{n} f(m) \leq \int_1^{n+1} f(x)dx$$

Using the following figure, we can see (on the left) that $\int_0^n f(x)dx$ underestimates the value of the summation while (on the right) $\int_1^{n+1} f(x)dx$ overestimates it.



83. Show that $\int_0^n 2x\,dx \leq \sum_{m=1}^{n} 2m \leq \int_1^{n+1} 2x\,dx$ (see Exercise 2).

84. Show that $\int_0^n x^2 dx \leq \sum_{m=1}^{n} m^2 \leq \int_1^{n+1} x^2 dx$ (see Exercise 7).