Although it is interesting and perhaps surprising to learn that there are uncountable sets, we are usually concerned with countable sets. A computer, of course, can manage only finite sets. In the rest of this chapter, we too, limit our attention to finite sets and various ways to count their elements.

SECTION 4.1 REVIEW

TECHNIQUES

- Describe sets by a list of elements and by a characterizing property.
- Prove that one set is a subset of another.
- Find the power set of a set.
- Check that the required properties for a binary or unary operation are satisfied.
- Form new sets by taking the union, intersection, complement, and cross product of sets.
- Prove set identities by showing set inclusion in each direction or using the basic set identities.
- Demonstrate the denumerability of certain sets.
- Use the Cantor diagonalization method to prove that certain sets are uncountable.

MAIN IDEAS

- Sets are unordered collections of objects that can be related (equal sets, subsets, etc.) or combined (union, intersection, etc.).
- Certain standard sets have their own notation.
- The power set of a set with n elements has 2^n elements.
- Basic set identities exist (in dual pairs) and can be used to prove other set identities; once an identity is proved in this manner, its dual is also true.
- Countable sets can be enumerated, and uncountable sets exist.

EXERCISES 4.1

1. Let $S = \{2, 5, 17, 27\}$. Which of the following expressions are true?
   a. $5 \in S$ b. $2 + 5 \in S$ c. $\varnothing \in S$ d. $S \in S$
2. Let $B = \{x \mid x \in \mathbb{Q} \text{ and } -1 < x < 2\}$. Which of the following expressions are true?
   a. $0 \in B$ b. $-1 \in B$ c. $-0.84 \in B$ d. $\sqrt{2} \in B$
3. How many different sets are described here? What are they?
   - $\{2, 3, 4\}$
   - $\varnothing$
   - $\{x \mid x$ is the first letter of cat, bat, or apple$\}$
   - $\{x \mid x$ is the first letter of cat, bat, and apple$\}$
   - $\{x \mid x \in \mathbb{N} \text{ and } 2 \le x \le 4\}$
   - $\{2, a, 3, b, 4, c\}$
   - $\{a, b, c\}$
   - $\{3, 4, 2\}$
4. How many different sets are described here? What are they?
   - $\{x \mid x = F(n) \wedge n \in \{5, 6, 7\}\}$ [$F(n)$ is a Fibonacci number]
   - $\{x \mid x \mid 24\}$ [x divides 24]
   - $\{1, 2, 3, 4\}$
   - $\{5, 8, 13\}$
   - $\{x \mid x \in \mathbb{N} \wedge 0 < x \le 4\}$
   - $\{x \mid x \in \varphi(5)\}$ [$\varphi(n)$ is the Euler phi function]
   - $\{12, 2, 6, 24, 8, 3, 1, 4\}$
   - $\{x \mid x$ is a digit in the decimal equivalent of the Roman numeral MCCXXXIV$\}$

5. Describe each of the following sets by listing its elements:
   a. $\{x \mid x \in \mathbb{N} \text{ and } x^2 < 25\}$
   b. $\{x \mid x \in \mathbb{N} \text{ and } x \text{ is even and } 2 < x < 11\}$
   c. $\{x \mid x$ is one of the first three U.S. presidents$\}$
   d. $\{x \mid x \in \mathbb{R} \text{ and } x^2 = -1\}$
   e. $\{x \mid x$ is one of the New England states$\}$
   f. $\{x \mid x \in \mathbb{Z} \text{ and } |x| < 4\}$ ($|x|$ denotes the absolute value function)
6. Describe each of the following sets by listing its elements:
   a. $\{x \mid x \in \mathbb{N} \text{ and } x^2 - 5x + 6 = 0\}$
   b. $\{x \mid x \in \mathbb{R} \text{ and } x^2 = 7\}$
   c. $\{x \mid x \in \mathbb{N} \text{ and } x^2 - 2x - 8 = 0\}$
7. Describe each of the following sets by giving a characterizing property:
   a. $\{1, 2, 3, 4, 5\}$
   b. $\{1, 3, 5, 7, 9, 11, \ldots\}$
   c. {Melchior, Gaspar, Balthazar}
   d. $\{0, 1, 10, 11, 100, 101, 110, 111, 1000, \ldots\}$
8. Describe each of the following sets:
   a $\{x \mid x \in \mathbb{N} \text{ and } (\exists q)(q \in \{2, 3\} \text{ and } x = 2q)\}$
   b. $\{x \mid x \in \mathbb{N} \text{ and } (\exists y)(\exists z)(y \in \{0, 1\} \text{ and } z \in \{3, 4\} \text{ and } y < x < z)\}$
   c. $\{x \mid x \in \mathbb{N} \text{ and } (\forall y)(y \text{ even} \rightarrow x \neq y)\}$
9. Given the description of a set A as $A = \{2, 4, 8 \ldots\}$, do you think $16 \in A$?
10. What is the cardinality of each of the following sets?
   a. $S = \{a, \{a, \{a\}\}\}$
   b. $S = \{\{a\}, \{\{a\}\}\}$
   c. $S = \{\emptyset\}$
   d. $S = \{a, \{\emptyset\}, \emptyset\}$
   e. $S = \{\emptyset, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset, \{\emptyset\}\}\}\}$
11. Let

$$A = \{2, 5, 7\}$$
$$B = \{1, 2, 4, 7, 8\}$$
$$C = \{7, 8\}$$

   Which of the following statements are true?

   a. $5 \subseteq A$ d. $7 \in B$
   b. $C \subseteq B$ e. $\{2, 5\} \subseteq A$
   c. $\emptyset \in A$ f. $\emptyset \subseteq C$
12. Let

$$A = \{x \mid x \in \mathbb{N} \text{ and } 1 < x < 50\}$$
$$B = \{x \mid x \in \mathbb{R} \text{ and } 1 < x < 50\}$$
$$C = \{x \mid x \in \mathbb{Z} \text{ and } |x| \geq 25\}$$

Which of the following statements are true?

a. $A \subseteq B$
b. $17 \in A$
c. $A \subseteq C$
d. $-40 \in C$
e. $\sqrt{3} \in B$
f. $\{0, 1, 2\} \subseteq A$
g. $\varnothing \in B$
h. $\{x \mid x \in \mathbb{Z} \text{ and } x^2 > 625\} \subseteq C$

13. Let

$$R = \{1, 3, \pi, 4.1, 9, 10\} \qquad T = \{1, 3, \pi\}$$
$$S = \{\{1\}, 3, 9, 10\} \qquad U = \{\{1, 3, \pi\}, 1\}$$

Which of the following statements are true? For those that are not, why not?

a. $S \subseteq R$
b. $1 \in R$
c. $1 \in S$
d. $1 \subseteq U$
e. $\{1\} \subseteq T$
f. $\{1\} \subseteq S$
g. $T \subset R$

14. Let

$$R = \{1, 3, \pi, 4.1, 9, 10\} \qquad T = \{1, 3, \pi\}$$
$$S = \{\{1\}, 3, 9, 10\} \qquad U = \{\{1, 3, \pi\}, 1\}$$

Which of the following statements are true? For those that are not, why not?

a. $\{1\} \in S$
b. $\varnothing \subseteq S$
c. $T \subseteq U$
d. $T \in U$
e. $T \notin R$
f. $T \subseteq R$
g. $S \subseteq \{1, 3, 9, 10\}$

15. Let

$$A = \{a, \{a\}, \{\{a\}\}\} \qquad B = \{a\} \qquad C = \{\varnothing, \{a, \{a\}\}\}$$

Which of the following statements are true? For those that are not, where do they fail?

a. $B \subseteq A$
b. $B \in A$
c. $C \subseteq A$
d. $\varnothing \subseteq C$
e. $\varnothing \in C$
f. $\{a, \{a\}\} \in A$
g. $\{a, \{a\}\} \subseteq A$
h. $B \subseteq C$
i. $\{\{a\}\} \subseteq A$

16. Let

$$A = \{\varnothing, \{\varnothing, \{\varnothing\}\}\} \qquad B = \varnothing \qquad C = \{\varnothing\} \qquad D = \{\varnothing, \{\varnothing\}\}$$

Which of the following statements are true? For those that are not, where do they fail?

a. $C \subseteq A$
b. $C \in A$
c. $B \in A$
d. $B \subseteq A$
e. $B \in C$
f. $C = B$
g. $C \subseteq D$
h. $C \in D$
i. $D \subseteq A$

17. Let

$$A = \{(x, y) \mid (x, y) \text{ lies within 3 units of the point } (1, 4)\}$$

and

$$B = \{(x, y) \mid (x - 1)^2 + (y - 4)^2 \leq 25\}$$

Prove that $A \subset B$.

18. Let

$$A = \{x \mid x \in \mathbb{R} \text{ and } x^2 - 4x + 3 < 0\}$$

and

$$B = \{x \mid x \in \mathbb{R} \text{ and } 0 < x < 6\}$$

Prove that $A \subset B$.

19. Program QUAD finds and prints solutions to quadratic equations of the form $ax^2 + bx + c = 0$. Program EVEN lists all the even integers from $-2n$ to $2n$. Let Q denote the set of values output by QUAD and E denote the set of values output by EVEN.
    a. Show that for $a = 1$, $b = -2$, $c = -24$, and $n = 50$, $Q \subseteq E$.
    b. Show that for the same values of a, b, and c, but a value for n of 2, $Q \not\subseteq E$.
20. Let $A = \{x \mid \cos(x/2) = 0\}$ and $B = \{x \mid \sin x = 0\}$. Prove that $A \subseteq B$.
21. Which of the following statements are true for all sets A, B, and C?
    a. If $A \subseteq B$ and $B \subseteq A$, then $A = B$.
    b. $\{\varnothing\} = \varnothing$
    c. $\{\varnothing\} = \{0\}$
    d. $\varnothing \in \{\varnothing\}$
    e. $\varnothing \subseteq A$
22. Which of the following statements are true for all sets A, B, and C?
    a. $\varnothing \in A$
    b. $\{\varnothing\} = \{\{\varnothing\}\}$
    c. If $A \subset B$ and $B \subseteq C$, then $A \subset C$.
    d. If $A \neq B$ and $B \neq C$, then $A \neq C$.
    e. If $A \in B$ and $B \not\subseteq C$, then $A \notin C$.
23. Prove that if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.
24. Prove that if $A' \subseteq B'$, then $B \subseteq A$.
25. Prove that for any integer $n \geq 2$, a set with n elements has $n(n - 1)/2$ subsets that contain exactly two elements.
26. Prove that for any integer $n \geq 3$, a set with n elements has $n(n - 1)(n - 2)/6$ subsets that contain exactly three elements. (*Hint*: Use Exercise 25.)
27. Find $\wp(S)$ for $S = \{a\}$.
28. Find $\wp(S)$ for $S = \{a, b\}$.
29. Find $\wp(S)$ for $S = \{1, 2, 3, 4\}$. How many elements do you expect this set to have?
30. Find $\wp(S)$ for $S = \{\varnothing\}$.

31. Find $\wp(S)$ for $S = \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}$.
32. Find $\wp(\wp(S))$ for $S = \{a, b\}$.
33. What can be said about A if $\wp(A) = \{\varnothing, \{x\}, \{y\}, \{x, y\}\}$?
34. What can be said about A if $\wp(A) = \{\varnothing, \{a\}, \{\{a\}\}\}$?
35. Prove that if $\wp(A) = \wp(B)$, then $A = B$.
36. Prove that if $A \subseteq B$, then $\wp(A) \subseteq \wp(B)$.
37. Solve for x and y.
    a. $(y, x + 2) = (5, 3)$ b. $(2x, y) = (16, 7)$ c. $(2x - y, x + y) = (-2, 5)$
38. a. Recall that ordered pairs must have the property that $(x, y) = (u, v)$ if and only if $x = u$ and $y = v$. Prove that $\{\{x\}, \{x, y\}\} = \{\{u\}, \{u, v\}\}$ if and only if $x = u$ and $y = v$. Therefore, although we know that $(x, y) \neq \{x, y\}$, we can define the ordered pair (x, y) as the set $\{\{x\}, \{x, y\}\}$.
    b. Show by an example that we cannot define the ordered triple (x, y, z) as the set $\{\{x\}, \{x, y\}, \{x, y, z\}\}$.
39. Which of the following candidates are binary or unary operations on the given sets? For those that are not, where do they fail?
    a. $x \circ y = x + 1$; $S = \mathbb{N}$
    b. $x \circ y = x + y - 1$; $S = \mathbb{N}$
    c. $x \circ y = \begin{cases} x - 1 & \text{if } x \text{ is odd} \\ x & \text{if } x \text{ is even} \end{cases}$ $S = \mathbb{Z}$
    d. $x\# = \ln x$; $S = \mathbb{R}$
40. Which of the following candidates are binary or unary operations on the given sets? For those that are not, where do they fail?
    a. $x\# = x^2$; $S = \mathbb{Z}$
    b.

∘	1	2	3
1	1	2	3
2	2	3	4
3	3	4	5

$S = \{1, 2, 3\}$

    c. $x \circ y$ = that fraction, x or y, with the smaller denominator; S = set of all fractions.
    d. $x \circ y$ = that person, x or y, whose name appears first in an alphabetical sort; S = set of 10 people with different names.
41. Which of the following candidates are binary or unary operations on the given sets? For those that are not, where do they fail?
    a. $x \circ y = \begin{cases} 1/x & \text{if } x \text{ is positive} \\ 1/(-x) & \text{if } x \text{ is negative} \end{cases}$ $S = \mathbb{R}$
    b. $x \circ y = xy$ (concatenation); S = set of all finite-length strings of symbols from the set $\{p, q, r\}$
    c. $x\# = \lfloor x \rfloor$ where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x; $S = \mathbb{R}$
    d. $x \circ y = \min(x, y)$; $S = \mathbb{N}$
42. Which of the following candidates are binary or unary operations on the given sets? For those that are not, where do they fail?
    a. $x \circ y$ = greatest common multiple of x and y; $S = \mathbb{N}$
    b. $x \circ y = x + y$; S = the set of Fibonacci numbers