1. (6 pts) Someone conjectured the following property of Fibonacci numbers for $n \geq 3$:

$$F(n+1) + F(n-2) = 2F(n)$$

$F(4) + F(1) = 2F(3)$
$3 + 1 = 4$

a. Verify that it works for $n = 3$, $n = 4$, and $n = 5$.

$n=3,\ F(3+1) + F(3-2) = 2F(3)$
$= F(4) + F(1) = 2F(3)$
$= 3 + 1 = 4$
$= 4 = \underline{\underline{4}}$

$n=5,\ F(5+1) + F(5-2) = 2F(5)$
$= F(6) + F(3) = 2F(5)$
$= 8 + 2 = 2(5)$
$= 10 = \underline{\underline{10}}$

$n=4,\ F(4+1) + F(4-2) = 2F(4)$
$= F(5) + F(2) = 2F(4)$
$= 5 + 1 = 6$
$6 = \underline{\underline{6}}$

b. Prove that it works for all $n \geq 3$.

$P(m): F(m+1) + F(m-2) = 2F(m)$

$P(m+1): F((m+1)+1) + F((m+1)-2) = 2F(m+1) : F(m+2) + F(m-1) = 2F(m+1)$

Assume for all k, $3 \leq k \leq m$

$P(m+1): F(m+2) + F(m-1) = F(m+1) + F(m) + F(m-1) = F(m+1) + F(m+1)$
$= 2F(m+1)$

$\therefore F(m+2) + F(m-1) = \underline{\underline{2F(m+1)}}$

$P(m) \rightarrow P(m+1)$

1. (6 pts) Someone conjectured the following property of Fibonacci numbers for $n \geq 3$:

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a. Verify that it works for $n = 3$, $n = 4$, and $n = 5$.

$n=3$, $F(3+1) + F(3-2) = 2F(3)$, $F(4) + F(1) = 2F(3)$, $3+1 = 2(2)$, $4=4$ ✓

$n=4$, $F(4+1) + F(4-2) = 2F(4)$, $F(5) + F(2) = 2F(4)$, $5+1 = 2(3)$, $6=6$ ✓

$n=5$, $F(5+1) + F(5-2) = 2F(5)$, $F(6) + F(3) = 2F(5)$, $8+2 = 2(5)$, $10=10$ ✓

✓

b. Prove that it works for all $n \geq 3$.

Base

Assume: $P(r)$, $3 \leq r \leq K$

~~F(K+1~~

Show: $F((K+1)+1) + F((K+1)-2) = 2F(K+1)$

$F(K+2) + F(K-1) = [F(K+1) + F(K)] + [F(K-2) + F(K-3)] = [F(K+1) + F(K-2)] + [F(K) + F(K-3)]$

$2F(K+1) = 2[F(K) + F(K-1)]$

$= 2F(K) + 2F(K-1) = 2[F(K) + F(K-1)]$

$= 2F(K+1)$ ✓

Good

1. (6 pts) Someone conjectured the following property of Fibonacci numbers for $n \geq 3$:

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a. Verify that it works for $n = 3$, $n = 4$, and $n = 5$.

$n = 3$: $F(n+1) + F(n-2) = F(4) + F(1) = 3 + 1 = 4$

$2F(n) = 2F(3) = 2 \cdot 2 = 4$

$\rightarrow$ Correct at $n = 3$

$n = 4$: $F(n+1) + F(n-2) = F(5) + F(2) = 5 + 1 = 6$

$2F(n) = 2F(4) = 2 \cdot 3 = 6$

$\rightarrow$ Correct at $n = 4$

$n = 5$: $F(n+1) + F(n-2) = F(6) + F(3) = 8 + 2 = 10$

$2F(n) = 2 \cdot F(5) = 2 \cdot 5 = 10$

$\rightarrow$ Correct at $n = 5$.

b. Prove that it works for all $n \geq 3$.

Assume correct at $n = r$, $1 \leq r \leq k$:

$F(r+1) + F(r-2) = 2F(r)$

$n = k+1$: $F(k+2) + F(k-1) \stackrel{?}{=} 2F(k+1)$

$r = k$: $F(k+1) + F(k-2) = 2F(k)$

$r = k-1$: $F(k) + F(k-3) = 2F(k-1)$

$$\begin{aligned}
2F(k+1) &= 2[F(k) + F(k-1)] \\
&= 2F(k) + 2F(k-1) \\
&= [F(k+1) + F(k-2)] + [F(k) + F(k-3)] \\
&= [F(k+1) + F(k)] + [F(k-2) + F(k-3)] \\
&= F(k+2) + F(k-1)
\end{aligned}$$

$\Rightarrow$ Correct at $n = k+1$

$\Rightarrow$ proved.

1. (6 pts) Someone conjectured the following property of Fibonacci numbers for $n \geq 3$:

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a. Verify that it works for $n = 3$, $n = 4$, and $n = 5$.

$n=3$ $\quad F(4)+F(1)=2F(3) \Rightarrow 3+1=2(2) \quad 4=4$ ✓

$n=4$ $\quad F(5)+F(2)=2F(4) \Rightarrow 5+1=2(3) \Rightarrow 6=6$ ✓

$n=5$ $\quad F(6)+F(3)=2F(5) \Rightarrow 8+2=2(5) \Rightarrow 10=10$ ✓

✓

b. Prove that it works for all $n \geq 3$.

$$\begin{aligned} F(n+1)+F(n-2) &= 2F(n) \\ &= F(n)+F(n) \\ &= F(n-2)+\underbrace{F(n-1)+F(n)} \\ &= F(n-2)+F(n+1) \end{aligned}$$

$F(n+1)+F(n-2) = F(n+1)+F(n-2)$ ✓ Ta-da :)

✓

1. (6 pts) Someone conjectured the following property of Fibonacci numbers for $n \geq 3$:

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a. Verify that it works for $n = 3$, $n = 4$, and $n = 5$.

$n=3$: $F(3+1) + F(3-2) = 2F(3)$

$3 + 1 = 2(2)$ $\quad 4=4$ ✓

$n=4$: $F(4+1) + F(4-2) = 2F(4)$

$5 + 1 = 2(3)$ $\quad 6=6$ ✓

$n=5$: $F(6) + F(3) = 2F(5)$

$8 + 2 = 2(5)$

$10 = 10$ ✓

b. Prove that it works for all $n \geq 3$.

$P(3)$: $4=4$ ✓

$P(4)$: $6=6$ ✓

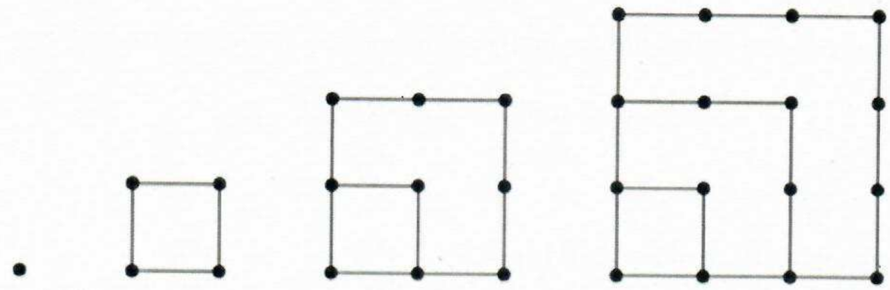
Assume $P(K)$ $P(r)$ is true for all r, $3 \leq r \leq K$

$P(K)$: $F(K+1) + F(K-2) = 2F(K)$

Show $P(K+1)$: $F(K+2) + F(K-1) = 2F(K+1)$

$$\begin{aligned} F(K+2) + F(K-1) &= [F(K+1) + F(K)] + [F(K+1) - F(K)] \\ &= [F(K+1) + (F(K-2) + F(K-1)] + [F(K+1) - F(K)] \\ &= [2F(K) + F(K-1)] + [F(K+1) - F(K)] \\ &= F(K) + F(K-1) + F(K+1) \\ &= F(K+1) + F(K+1) \\ &= 2F(K+1) \leftarrow \text{rhs of } P(K+1) \end{aligned}$$ ✓

2. (4 pts) Consider the first four in a sequence of dot diagrams, $S(1)$ through $S(4)$, each building off the previous dot diagram:



1 4 9 16

$S(n) = n^2$

Write and solve a recurrence relation for $S(n)$, the number of dots in the n^{th} diagram.

Reminder: here's the formula that gives the general solution for a first-order, linear, non-homogeneous recurrence relation:

$$S(n) = c^{n-1}S(1) + \sum_{i=2}^{n} c^{n-i}g(i)$$

$S(1) = 1$

$S(n) = S(n-1) + (2n-1)$

Note: $\sum_{i=1}^{n} 2i-1 = n^2$ as proven on last exam

$$S(n) = 1^{n-1}(1) + \sum_{i=2}^{n} 1^{n-i}(2i-1)$$

$$= 1 + \sum_{i=2}^{n}(2i-1)$$

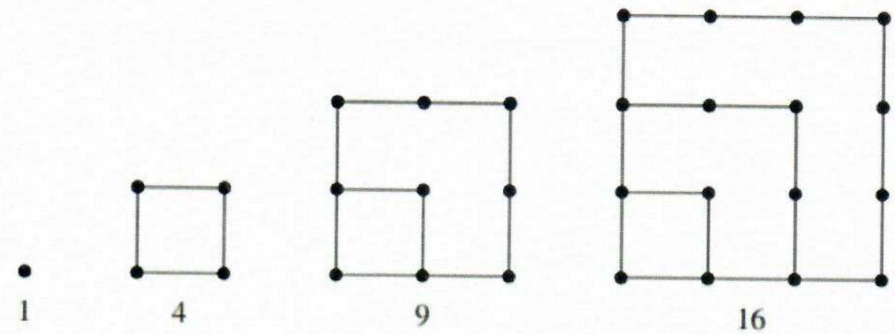
$$= 1 + \sum_{i=1}^{n}((2i-1)) - 1$$

$$= \sum_{i=1}^{n}(2i-1)$$

$$= n^2$$

Well done! Hooray!

2. (4 pts) Consider the first four in a sequence of dot diagrams, $S(1)$ through $S(4)$, each building off the previous dot diagram:



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Reminder: here's the formula that gives the general solution for a first-order, linear, non-homogeneous recurrence relation:

$$S(n) = c^{n-1}S(1) + \sum_{i=2}^{n} c^{n-i}g(i)$$

$$\begin{cases} S(1) = 1 \\ S(n) = S(n-1) + (n-1) + n = S(n-1) + 2n - 1 \end{cases}$$

n	$S(n)$
1	1
2	4
3	9
4	16
5	25

$$S(n) = (1)^{n-1}(1) + \sum_{i=2}^{n} (1)^{n-i}(2n-1)$$

$$\sum_{i=2}^{n} (1)^{n-i}(2n-1) = n^2 - 1$$

$$= 1 + (n^2 - 1)$$

$$= n^2$$

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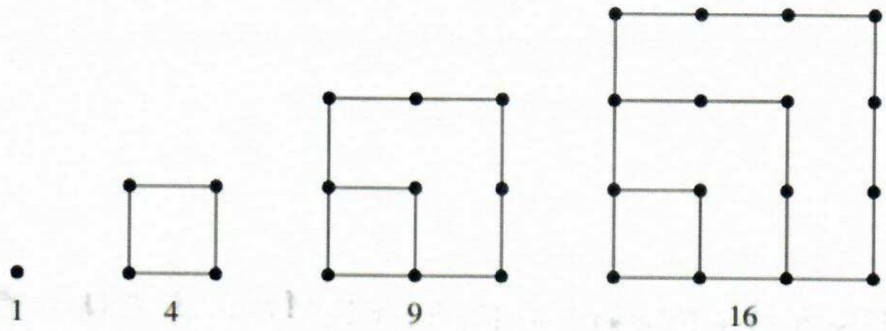

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Reminder: here's the formula that gives the general solution for a first-order, linear, non-homogeneous recurrence relation:

$$S(n) = c^{n-1}S(1) + \sum_{i=2}^{n} c^{n-i}g(i)$$

$S(1) = 1$

$S(2) = 4 = 1 + 3 = S(1) + 3 = S(1) + (2.2 - 1)$

$S(3) = 9 = 4 + 5 = S(2) + 5 = S(2) + (2.3 - 1)$

$S(4) = 16 = 9 + 7 = S(3) + 7 = S(3) + \cancel{4}(2.4 - 1)$

.....

$S(n) = S(n-1) + (2n-1)$

⇒ Recurrence relation :

$S(1) = 1$

$S(n) = S(n-1) + \cancel{(2n-1)}\ (2n-1)$ for $n \geq 2$

Good

$c = 1$

$g(n) = 2n - 1$

$$S(n) = c^{n-1} S(1) + \sum_{i=2}^{n} c^{n-i} g(i)$$

$$= 1^{n-1} \cdot 1 + \sum_{i=2}^{n} 1^{n-i} (2i-1)$$

$$= 1 + \sum_{i=2}^{n} (2i-1)$$

$$= 1 + [3 + 5 + 7 + \dots + (2n-1)]$$

$$= n^2$$

proved that last week!

⇒ Close-form solution for $S(n)$ is $S(n) = n^2$.