MAT385 Final, Spring 2020

Name:

Directions:

- Each problem is worth 10 points. **You must skip one of the "old" problems (1-5), and one of the new problems, 6-10.** Write "SKIP" clearly on each problem you skip. So you'll be doing 8 problems. You have two hours.
- Show your work! Answers without justification will likely result in few points. Your written work also allows me the option of giving you partial credit in the event of an incorrect final answer (but good reasoning). Indicate clearly your answer to each problem (e.g., put a box around it).
- **Good luck, thanks for sticking with me, and have a wonderful summer in spite of damned viruses.**

Problem 1: . Prove the following theorems (for parts a. and b., first write them as wffs):

a. Propositional logic (E,Q,B): If it is an Elephant, then it is Quiet. It is an Elephant or a Bear. It is not Quiet. Therefore it is a Bear.

b. Predicate logic (P,Q): There is someone who is both Peaceful and Quiet. Everyone Quiet is Reserved. Therefore there is someone both Quiet and Reserved.

c. Use TautologyTest to prove that $(A \wedge B)' \vee B' \longrightarrow A' \vee B'$

Problem 2: . (More!) Proofs:

a. Prove that the sum of a natural number and its square is even: $n + n^2$ is even for $n \in \mathbb{N}$.

b. Prove (you have options!) that $F(n+6) = 4F(n+3) + F(n)$ for $n \geq 1$, where $F(n)$ is the n^{th} Fibonacci number.

Problem 3: . In spite of what some politicians say, Covid-19 tests are still very hard to get.

To test blood from blood donors for coronavirus, small samples of blood from $n = 2^m$ ($m \in \mathbb{N}$) donors are pooled, and then **the pooled sample is tested**. If negative, great – all clear, after just one test; if positive, however, apply the strategy recursively on pooled samples of two halves, one after the other.

a. (5 pts) **Best case scenario**: Suppose we know that **only one person** is infected (**and the lab knows it, too**): how many Covid-19 tests will the lab do in order to identify the individual? (You might consider some simple cases.) How does that compare to just testing everyone?

b. (5 pts) **Worst case scenario**: Suppose we know that **all** are infected (**but the lab doesn't know it**). How many tests will the lab do in order to determine that? How does that compare to just testing everyone?

c. (**2 pts, extra credit**) **Reality**: is generally somewhere between best and worst cases. At what prevalence (infection rate) will sequential testing of all donors require about the same number of tests as this divide-and-conquer strategy? (You have to be within a tenth, with some reason...:)

Problem 4: Sets

a. Prove that the set of positive, even integers $E = \{2n | n \in \mathbb{N}\}$ has the same cardinality as the set of the positive powers of 2, $P = \{2^n | n \in \mathbb{N}\}$. (They are exactly the **same size**, as infinite sets.)

b. Let $R = \{1, 3, \pi, 4.1, 9, 10\}$, $S = \{\{1\}, 3, 9, 10\}$, $T = \{1, 3, \pi\}$, and $U = \{\{1, 3, \pi\}, 1\}$.

Which of the following statements are true? For those that are not, why not?

i. $\{1\} \in S$

ii. $\emptyset \subseteq S$

iii. $T \subset U$

iv. $T \in U$

v. $T \notin R$

vi. $T \subseteq R$

vii. $S \subseteq \{1, 3, 9, 10\}$

viii. $R = \{3, 1, 4.1, \pi, 10, 9\}$

ix. $R \subset R$

x. $\emptyset \in R$

Problem 5:

a. (2 pts) Draw K_3.

i. (3 pts) How many simple subgraphs of K_3 are there? Draw them all.

ii. (3 pts) Label the arcs of K_3. Show how **each** simple graph with three vertices can be associated with a **subset** of the power set of the set of arcs, in a sensible way.

b. (2 pts) Suppose that the three vertices represent three different people (label them), and the arcs represent "friendship" (label them). How many distinctly different "friendship" graphs are possible? How are they related to the power set above?

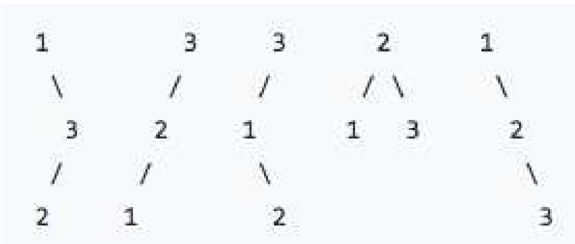
Problem 6: Trees

a. Draw the expression tree for (3*(2-5)+14)-((7+3)/(x*y)); then give preorder and postorder traversals.

i. Preorder:

ii. Postorder:

b. Given $n \in \mathbb{N}$, find $s(n)$ – the number of structurally unique binary search trees that store values 1 through n? Use recursion! (For convenience I defined $s(0) = 1$.) If you can't find a formula, draw all such distinct trees that store 1 through 4 for some credit. This figure shows that $s(3) = 5$.



Problem 7:

a. (3 pts) For the following truth table create a canonical sum-of-products Boolean expression for the truth function $f(x_1, x_2, x_3, x_4)$:

x_1	x_2	x_3	x_4	$f(x_1, x_2, x_3, x_4)$
1	1	1	1	1
1	1	1	0	1
1	1	0	1	1
1	1	0	0	0
1	0	1	1	0
1	0	1	0	1
1	0	0	1	0
1	0	0	0	0
0	1	1	1	1
0	1	1	0	1
0	1	0	1	1
0	1	0	0	1
0	0	1	1	0
0	0	1	0	1
0	0	0	1	0
0	0	0	0	0

b. (4 pts) Convert to the Karnaugh map for the canonical sum-of-products, and illustrate how to minimize the expression using the map. Write the minimized sum-of product here.

	x_1x_2	x_1x_2'	$x_1'x_2'$	$x_1'x_2$
x_3x_4				
x_3x_4'				
$x_3'x_4'$				
$x_3'x_4$				

c. (3 pts) Draw the logic network corresponding to this simplified expression.

Problem 8:

a. (6 pts) Prove that for any Boolean algebra $x \cdot y' = 0$ if and only if $x \cdot y = x$.

b. (4 pts) Interpret this result – $x \cdot y' = 0$ if and only if $x \cdot y = x$ – in the realm of

i. Propositional logic

ii. Set theory

Problem 9: In Problem 7 you used Karnaugh to simplify the truth function $f(x_1, x_2, x_3, x_4)$. Now use Quine-McCluskey to do the same. I've provided some tables to help you **get started** (but not finished!).

x_1	x_2	x_3	x_4	f
1	1	1	1	1
1	1	1	0	1
1	1	0	1	1
1	1	0	0	0
1	0	1	1	0
1	0	1	0	1
1	0	0	1	0
1	0	0	0	0
0	1	1	1	1
0	1	1	0	1
0	1	0	1	1
0	1	0	0	1
0	0	1	1	0
0	0	1	0	1
0	0	0	1	0
0	0	0	0	0

#1s	x_1 x_2 x_3 x_4
4	
3	
2	
1	

#1s	x_1 x_2 x_3 x_4
3	
2	
1	

Problem 10:

a. (3 pts) Write a regular expression for all strings which recognizes only strings that start with 1 and end with 101.

b. (5 pts) Build a finite state machine that recognizes such strings.

c. Illustrate your machine's operation on the string 10001101 using this table (recognition output is 1; otherwise 0):

Input	1	0	0	0	1	1	0	1	
State									
Output									