Directions: Problems are worth 20 points each. You must do problems 1 and 2; then you should choose three more to do from the remaining five (that is, problems 3-7). That is, you must skip two other complete problems (write "skip" clearly on the two you skip).

Show your work: answers without justification will likely result in few points. Your written work also allows me the option of giving you partial credit in the event of an incorrect final answer (but good reasoning). Indicate clearly your answers to each problem. **Good luck!**

Problem 1 (You may not skip this problem): Prove the following for all natural numbers $n \geq 1$:

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \ldots + \frac{1}{n \cdot (n+1)} = \frac{n}{n+1}$$

$P(n)$: $\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$ ✓

Base case : $P(1)$: $\frac{1}{2} = \frac{1}{1+1} \to P(1)$ true ✓

Induction step : $P(k) \to P(k+1)$

Assume $P(k)$: $\sum_{i=1}^{k} \frac{1}{\cancel{k(k+1)}\ i(i+1)} = \frac{k}{k+1}$

consider $P(k+1)$: $\sum_{i=1}^{k+1} \frac{1}{i(i+1)} \stackrel{?}{=} \frac{k+1}{k+2}$

LHS of $P(k+1)$: $\sum_{i=1}^{k+1} \frac{1}{i(i+1)} = \sum_{i=1}^{k} \frac{1}{i(i+1)} + \frac{1}{(k+1)(k+2)}$

$$= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} = \frac{k(k+2)+1}{(k+1)(k+2)}$$

$$= \frac{k^2+2k+1}{(k+1)(k+2)} = \frac{(k+1)^2}{(k+1)(k+2)} = \frac{k+1}{k+2} \to P(k+1) \text{ true}$$

$\therefore (\forall n)\, P(n)$ by 1st principle of mathematical induction ✓

Well done.

20

Problem 2 (You may not skip this problem): Consider the following wff:

$$(\forall x)(P(x) \vee Q(x)) \rightarrow ((\forall x)P(x)' \rightarrow (\forall x)Q(x)])$$

Prove this theorem in two ways:

a. directly

1. $(\forall x)(P(x) \vee Q(x))$ hyp
2. $(\forall x)P(x)'$ deduction method
3. $P(x) \vee Q(x)$ 1, ui
4. $P(x)'$ 2, ui
5. $Q(x)$ 3, 4, disjunctive syll.
6. $(\forall x)Q(x)$ 5, ug

b. by contraposition or contradiction (your choice – check the appropriate box).

☐ Contradiction

☑ Contraposition

$((\forall x)P(x)' \wedge (\exists x)Q(x)') \rightarrow (\exists x)(P(x)' \wedge Q'(x))$

1. $((\forall x)P(x)' \wedge (\exists x)Q(x)')$ hype
2. $(\forall x)P(x)'$ 1, simp
3. $(\exists x)Q(x)'$ 1, simp
4. $Q(x)'$ 3, ei
5. $P(x)'$ 2, ui
6. $P(x)' \wedge Q'(x)$ 4, 5, conj.
7. $(\exists x)(P(x)' \wedge Q'(x))$ 6, eg

Problem 2 (You may not skip this problem): Consider the following wff:

$$(\forall x)(P(x) \vee Q(x)) \rightarrow ((\forall x)P(x)' \rightarrow (\forall x)Q(x)])$$

Prove this theorem in two ways:

a. directly

1. $(\forall x)(P(x) \vee Q(x))$ --- hyp
2. $(\forall x)P(x)'$ hyp (ded)
3. $P(x) \vee Q(x)$ 1, ui
4. $P(x)'$ 2, ui
5. $Q(x)$ 3, 4, ds
6. $(\forall x)Q(x)$ 5, ug

b. by contraposition or contradiction (your choice – check the appropriate box).

☒ Contradiction $((\forall x)P(x)' \rightarrow (\forall x)Q(x))'$

☐ Contraposition $(\forall x)P(x)' \wedge (\exists x)Q(x)'$ good

1. $(\forall x)(P(x) \vee Q(x))$ hyp
2. $(\forall x)P(x)' \wedge (\exists x)Q(x)'$ hyp (contradiction)
3. $(\forall x)P(x)'$ 2, simp
4. $(\exists x)Q(x)'$ 2, simp
5. $Q(x)'$ 4, ei
6. $P(x)'$ 3, ui
7. $P(x) \vee Q(x)$ 1, ui
8. $P(x)' \wedge Q(x)'$ 5, 6, conj.
9. $(P(x) \vee Q(x))'$ 8, de morgan
10. $(P(x) \vee Q(x)) + (P(x) \vee Q(x))' \rightarrow 0$, 7, 9, inconsistency

∴ wff is valid by contradiction.

Well done!

20

Problem 3:

a. Translate the following into symbolic notation, using the statement letters B: the heroine is British; F: the villain is French; M: the Movie is good.

i. A British heroine is a necessary condition for the movie to be good.

$M \to B$

ii. The heroine is not British, but the villain is French.

$B' \wedge F$

b. Use B, F, and M as above to translate the following into English:

i. $B \to F'$

If the heroine is British, then the villain is not French.

ii. $(B \vee F') \to M$

If the heroine is British or the villain is not French, then the movie is good.

c. Use a truth table to establish the principle of contraposition:

$$(P \to Q) \leftrightarrow (Q' \to P')$$

P	Q	$P \to Q$	Q'	P'	$Q' \to P'$	$(P \to Q) \leftrightarrow (Q' \to P')$	
T	T	T	F	F	T	T	
T	F	F	T	F	F	T	
F	T	T	F	T	T	T	
F	F	T	T	T	T	T	

Problem 4:

a. Using the following symbols, write each English language statement as a predicate wff: $A(x)$ – x is an animal; $B(x)$ – x is a bear; $W(x)$ – x is a wolf; $H(x)$ – x is hungry.

i. If all wolves are hungry, so are bears.

$(\forall x)[W(x) \rightarrow H(x)] \rightarrow (\forall y)[B(y) \rightarrow H(y)]$

good work.

ii. Some wolves are hungry, but not every animal is hungry.

$(\exists x)[W(x) \wedge H(x)] \wedge (\exists y)[A(y) \wedge (H(y))']$

b. Decide whether this wff is valid or invalid. Justify your answer.

$(\forall x)P(x) \vee (\exists x)Q(x) \rightarrow (\forall x)(P(x) \vee Q(x))$

This is invalid. Take $(\exists x)Q(x)$ in the antecedent to be true and $(\forall x)P(x)$ to be false. All $(\exists x)Q(x)$ says is that there is an x that has the characteristic Q. The universe is still open to many different possibilities for x other than P or Q.

c. Using predicate logic, and the predicate symbols provided, prove that the following argument is valid. Every teacher lectures only to students, and some teacher lectures to someone. Therefore, there is a student. T(x), L(x,y), S(x)

$(\forall x)[T(x) \rightarrow (\forall y)(L(x,y) \rightarrow S(y))] \wedge (\exists x)(\exists y)[T(x) \wedge L(x,y)] \longrightarrow (\exists x)S(x)$

1. $(\forall x)(\forall y)[T(x) \rightarrow (L(x,y) \rightarrow S(y))]$ hyp
2. $(\exists x)(\exists y)[T(x) \wedge L(x,y)]$ hyp
3. $(\exists x)[T(x) \wedge L(x,b)]$ 2, ei
4. $T(a) \wedge L(a,b)$ 3, ei
5. $(\forall x)[T(x) \rightarrow (L(x,b) \rightarrow S(b))]$ 1, ui
6. $T(a) \rightarrow (L(a,b) \rightarrow S(b))$ 5, ui
7. $T(a)$ 4, sim
8. $L(a,b)$ 4, sim
9. $L(a,b) \rightarrow S(b)$ 7, 6, mp
10. $S(b)$ 8, 9, mp
11. $(\exists x)S(x)$ 10, eg

Great

20

Problem 5: A sequence is recursively defined by

$$\begin{aligned} T(0) &= 1 \\ T(1) &= 2 \\ T(n) &= 2T(n-1)+T(n-2) \quad n \geq 2 \end{aligned}$$

Prove that $T(n) \leq \left(\frac{5}{2}\right)^n$ for $n \geq 2$.

$P(2): 2T(2-1) + T(2-2) \leq \left(\frac{5}{2}\right)^2$

$2(2) + 1 \leq \frac{25}{4} \qquad 5 \leq 6.25$ ✓

$P(3): 2T(3-1) + T(3-2) \leq \left(\frac{5}{2}\right)^3$

$2(5) + 2 \leq \frac{125}{8} \qquad 12 \leq \frac{125}{8}$ ✓

No, you're good!

Assume $P(r): 2T(r-1) + T(r-2) \leq \left(\frac{5}{2}\right)^r$ for all r, $4 \leq r \leq k$

$P(k+1): T(k+1) \leq \left(\frac{5}{2}\right)^{k+1}$

$T(k+1) = 2T(k) + T(k-1) \leq 2\left(\frac{5}{2}\right)^k + \left(\frac{5}{2}\right)^{k-1} = \left(\frac{5}{2}\right)^k \left(2 + \left(\frac{5}{2}\right)^{-1}\right)$

$= \left(\frac{5}{2}\right)^k (2.4) \leq \left(\frac{5}{2}\right)^k \left(\frac{5}{2}\right) = \left(\frac{5}{2}\right)^{k+1}$ ✓

rhs of $P(k+1)$

Nice work

Problem 5: A sequence is recursively defined by

$$T(0) = 1$$
$$T(1) = 2$$
$$T(n) = 2T(n-1) + T(n-2) \quad n \geq 2$$

Prove that $T(n) \leq \left(\frac{5}{2}\right)^n$ for $n \geq 2$.

$n = 0$: $T(n) = T(0) = 1$

$\left(\frac{5}{2}\right)^n = \left(\frac{5}{2}\right)^0 = 1$ ✓

$\Rightarrow$ Correct at $n = 0$

$n = 1$: $T(n) = T(1) = 2$

$\left(\frac{5}{2}\right)^n = \left(\frac{5}{2}\right)^1 = 2.5$ ✓

$\Rightarrow$ Correct at $n = 1$

Assume correct at $n = r$, $1 \leq r \leq k$:

$T(r) \leq \left(\frac{5}{2}\right)^r$ ✓

Prove at $n = k+1$:

$T(k+1) \leq \left(\frac{5}{2}\right)^{k+1}$

Well done

$r = k$: $T(k) \leq \left(\frac{5}{2}\right)^k$

$r = k-1$: $T(k-1) \leq \left(\frac{5}{2}\right)^{k-1}$

$$T(k+1) = 2T(k) + T(k-1)$$
$$\leq 2\left(\frac{5}{2}\right)^k + \left(\frac{5}{2}\right)^{k-1}$$
$$\cancel{= \left(\frac{5}{2}\right)^{k-1}(}$$
$$= 2 \cdot \left(\frac{5}{2}\right)^{k-1} \cdot \frac{5}{2} + \left(\frac{5}{2}\right)^{k-1}$$
$$= \left(\frac{5}{2}\right)^{k-1}\left(2 \cdot \frac{5}{2} + 1\right)$$
$$= \left(\frac{5}{2}\right)^{k-1} \cdot 6$$
$$\leq \left(\frac{5}{2}\right)^{k-1} \cdot \left(\frac{5}{2}\right)^2 = \left(\frac{5}{2}\right)^{k+1}$$

$\Rightarrow$ Correct at $n = k+1$

$\Rightarrow$ Proved.

20

Problem 6: Solve the recurrence relation subject to the basis step:

$$P(1) = 1$$
$$P(n) = 2P\left(\frac{n}{2}\right) + n^2 \quad n \geq 2,\ n = 2^m$$

$c = 2$, $g(n) = n^2$

$$P(n) = c^{\log n} P(1) + \sum_{i=1}^{\log n} c^{\log n - i} g(2^i)$$

$$= 2^{\log n} \cdot 1 + \sum_{i=1}^{\log n} 2^{\log n - i} (2^i)^2$$

$$= n + \sum_{i=1}^{\log n} 2^{\log n - i} \cdot 2^{2i}$$

$$= n + \sum_{i=1}^{\log n} 2^{\log n + i}$$

$$= n + \sum_{i=1}^{\log n} 2^{\log n} \cdot 2^i$$

$$= n + \sum_{i=1}^{\log n} n \cdot 2^i$$

$$= n + n \sum_{i=1}^{\log n} 2^i$$

$$= n(1 + 2^1 + 2^2 + 2^3 + \dots + 2^{\log n})$$

$$= n(2^{\log n + 1} - 1)$$

$$= n(2^{\log n} \cdot 2 - 1)$$

$$= n(n \cdot 2 - 1)$$

$$= 2n^2 - n$$

$\Rightarrow$ Close-form of $P(n)$: $P(n) = 2n^2 - n$

Problem 7: The next time you sort a deck of cards, you should try it by a mergesort. There are four suits, ordered clubs, diamonds, hearts, spades; each suit has 13 cards, 1 to 13. But, for our purposes, you can think of the cards being numbered 1-52.

What is a reasonable bound on the worst case number of comparisons required to mergesort the deck (e.g. since it's just an upper bound, you might consider 64 cards, rather than 52). **You will need to justify your answer by some analysis**, using the details of the mergesort algorithm – basically cutting a list in half, mergesorting each half, and then merging the two halves together (do you remember the worst case for a merge of two sorted lists of length r and s?).

Hint: you might start from the bottom: how many comparisons to sort a list with 1 element?

$C(1) = 0$

$C(n) = 2C(\frac{n}{2}) + n - 1$

$C(n) = 2^m \cdot 0 + \sum_{i=1}^{m} 2^{m-i} \cdot (2^i - 1)$

$= \sum_{i=1}^{m} 2^m - \sum_{i=1}^{m} 2^{m-i}$

$= m 2^m - \frac{2^m - 1}{2-1}$

$= m 2^m - 2^m + 1$

$= n(\log n - 1) + 1$

$C(52) < C(64) = 64(6-1) + 1$

$= 321$

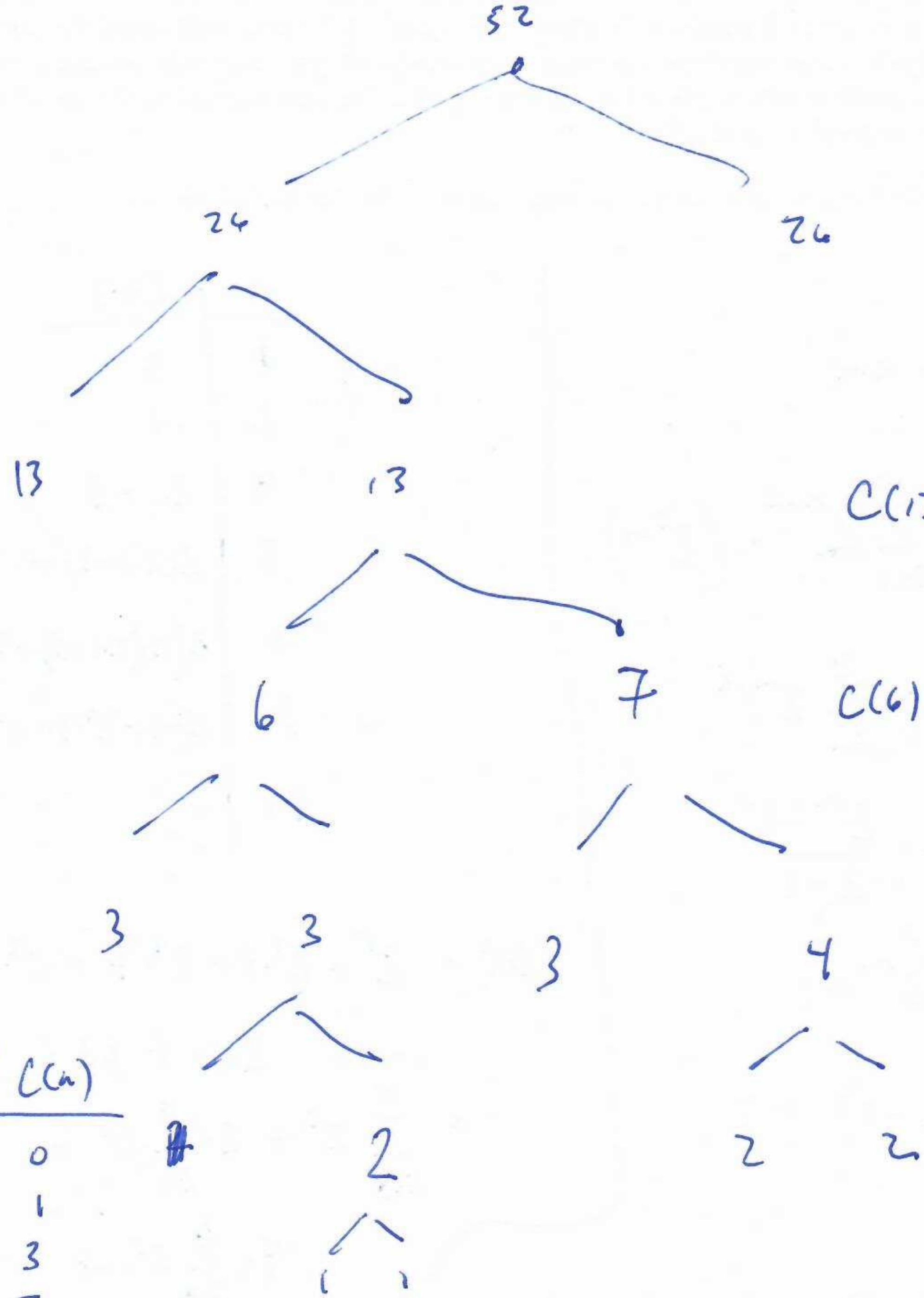
n	$C(n)$
1	0
2	1
4	$2 \cdot 1 + 3$
8	$2(2 \cdot 1 + 3) + 7$
16	$2(2 \cdot (2 \cdot 1 + 3) + 7) + 15$
32	$2^4 \cdot 1 + 2^3 \cdot 3 + 2^2 \cdot 7 + 2 \cdot 15 + 31$
64	

$C(64) = 2^5 + 2^4 \cdot 3 + 2^3 \cdot 7 + 2^2 \cdot 15 + 2 \cdot 31 + 63$

$= \sum_{i=0}^{5} 2^i + 2 \cdot \sum_{i=0}^{4} 2^i + 4 \cdot \sum_{i=0}^{3} 2^i + 8 \cdot \sum_{i=0}^{2} 2^i + 16 \cdot \sum_{i=0}^{1} 2^i$

$(2^6 - 1) + 2(2^5 - 1) + 2^2(2^4 - 1) + 2^3(2^3 - 1) + 2^4(2^2 - 1) + 2^5(2^1 - 1)$

$6 \cdot 2^6 - (2^6 - 1) = 5 \cdot 2^6 + 1 = 321$



$C(52) = 2C(26) + 51$

$C(26) = 2C(13) + 25$

$C(13) = C(6) + C(7) + 12$

$C(6) = 2C(3) + 5$

$C(7) = C(3) + C(4) + 6$

$C(4) = 2C(2) + 3$

$C(2) = 1$

$C(3) = C(1) + C(2) + 2$

$C(1) = 0$

n	$C(n)$
1	0
2	1
3	3
4	5
6	11
7	14
13	37
26	99
52	$\boxed{249}$