In the higher degree system, only the first three of these ten characters are used, namely *yì*, *zhāo* and *jing*. These are given the values 10^8, 10^{16} and 10^{32} respectively. With these, it is possible to represent all numbers less than 10^{64}. For example:

三京五千三百一億二百七萬六千一百八十五兆三億一萬

sān jing wǔ qiān sān bǎi yī yì èr bǎi qī wàn liù qiān yī bǎi bā shí wǔ zhāo sān yì yī wàn

$(3 \times 10^{32} + [[5 \times 10^3 + 3 \times 10^2 + 1] . 10^8 + [2 \times 10^2 + 7] \cdot 10^4 + 6 \times 10^3 + 1 \times 10^2 + 8 \times 10 + 5] \cdot 10^{16} + 3 \times 10^8 + 1 \times 10^4$
300,005,301,020,761,850,000,000,300,010,000

FIG. 21.41.

		Xià deng LOWER DEGREE SYSTEM	*Zhōng deng* MIDDLE DEGREE SYSTEM	*Shàng deng* HIGHER DEGREE SYSTEM
萬	*wàn*	10^4	10^4	10^4
億	*yì*[a]	10^5	10^8	10^8
兆	*zhāo*	10^6	10^{12}	10^{16}
京	*jing*	10^7	10^{16}	10^{32}
垓	*gai*	10^8	10^{20}	10^{64} (THEORETICAL VALUES)
補	*bù*[b]	10^9	10^{24}	10^{128}
壤	*ràng*	10^{10}	10^{28}	10^{256}
冓	*gou*[c]	10^{11}	10^{32}	10^{512}
澗	*jián*	10^{12}	10^{36}	10^{1024}
正	*zheng*	10^{13}	10^{40}	10^{2048}
載	*zái*	10^{14}	10^{44}	10^{4096}

[a] Graphical variant 亿 [b] Equivalent word 溝 [c] Graphical variant 秭

FIG. 21.42. *Chinese scientific notation for large numbers [Giles (1912); Mathews (1931); Needham (1959)]*

Such very large numbers are, however, very infrequently used: "in mathematics, business or economics numbers greater than 10^{14} are very rare; only in connection with astronomy or the calendar do we sometimes find larger numbers" [R. Schrimpf (1963–64)].

Finally, let us draw attention to a very interesting notation which Chinese and Japanese scientists have used to express negative powers of 10:

$$10^{-1} = 1/10,\ 10^{-2} = 1/100,\ 10^{-3} = 1/1{,}000,\ 10^{-4} = 10{,}000, \text{ etc.}$$

They especially find mention in the arithmetical treatise *Jinkoki* published in 1627 by the Japanese mathematician Yoshida Mitsuyoshi (Fig. 21.43).

分	*fēn*	10^{-1}
厘	*lí*	10^{-2}
毛	*máo*	10^{-3}
糸	*mi*	10^{-4}
忽	*hū*	10^{-5}
微	*wēi*	10^{-6}
繊	*xiān*	10^{-7}
沙	*shā*	10^{-8}
塵	*chén*	10^{-9}
埃	*āi*	10^{-10}

FIG. 21.43. *Sino-Japanese scientific notation for negative powers of 10 [Yamamoto (1985)]*

THE CHINESE SCIENTIFIC POSITIONAL SYSTEM

Further evidence of advanced intellectual development in the Far East comes from the written positional notation formerly used by Chinese, Japanese, and Korean mathematicians.

Though we only know examples of this system dating back to the second century BCE, it seems probable that it goes back much further.

Known by the Chinese name *suan zí* (literally, "calculation with rods"), and by the Japanese name *sangi*, this system is similar to our modern number-system not only by virtue of its decimal base, but also because the values of the numerals are determined by the position they occupy. It is therefore a strictly positional decimal number-system.

However, whereas our system uses nine numerals whose forms carry no intrinsic suggestion of value, this system of numerals makes use of systematic combinations of horizontal and vertical bars to represent the first nine units. The symbols for 1 to 5 use a corresponding number of vertical strokes, side by side, and the symbols for 6, 7, 8, and 9 show a horizontal bar capping 1, 2, 3, or 4 vertical strokes:

FIG. 21.44.

Examples of numbers written in this system are given by Cai Jiu Feng, a Chinese philosopher of the Song era who died in 1230 [in *Huang ji*, in the chapter *Hong fan* of his "Book of Annals", cited by A.Vissière (1892)]. Example:

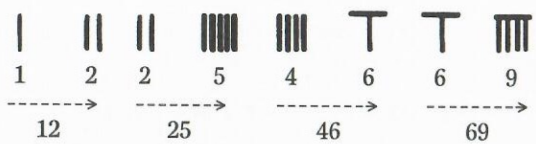

FIG. 21.45.

Ingenious as it was, this system lent itself to ambiguity.

For one thing, people writing in this system tended to place the vertical bars for the different orders of magnitude side by side. So the notation for the number 12 could be confused with that for 3 or for 21; 25 could be confused with 7, 34, 43, 52, 214, or 223, and so on (Fig. 21.45).

However, the Chinese found a way round the problem, by introducing a second system for the units, analogous to the first but made up of horizontal bars rather than vertical. The first five digits were represented by as many horizontal bars, and the numbers 6, 7, 8, 9 by erecting a vertical bar (with symbolic value 5) on top of one, two, three, or four horizontal bars:

FIG. 21.46.

Then, to distinguish between one order of magnitude and the next, they alternated figures from one series with figures from the other, therefore alternately vertical and horizontal. The units, hundreds, tens of thousands, millions, and so on (of odd rank) were drawn with "vertical" symbols (Fig. 21.44), whereas the tens, thousands, hundreds of thousands, tens of millions, etc. (of even rank) were drawn with "horizontal" symbols (Fig. 21.46), by which means the ambiguities were elegantly resolved (Fig. 21.48).

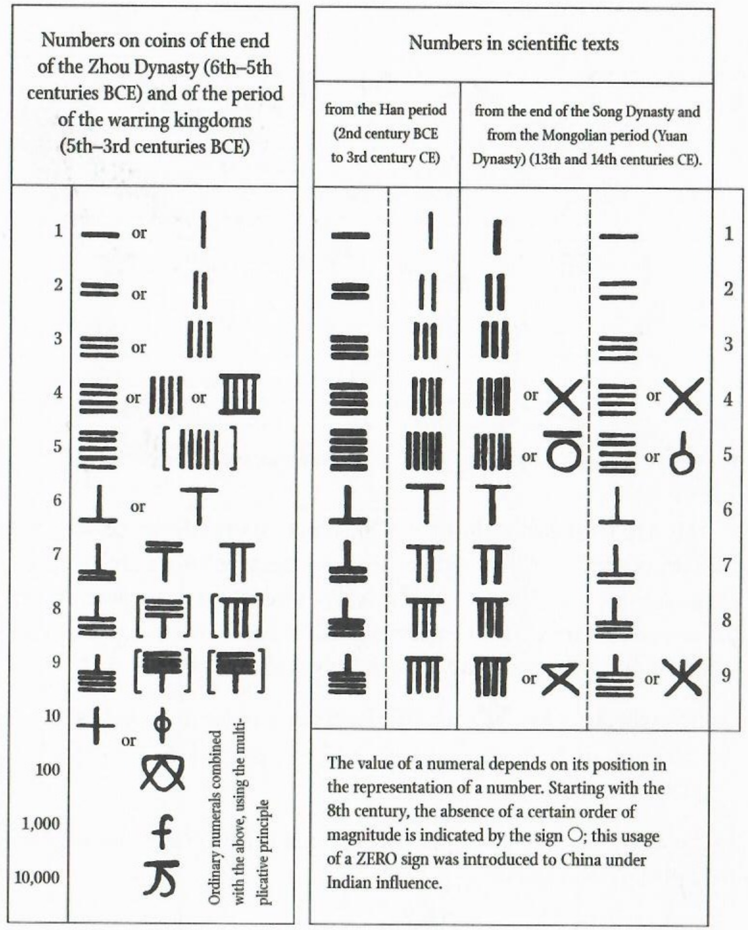

FIG. 21.47. *Chinese bar numerals through the ages [Needham (1959)]*

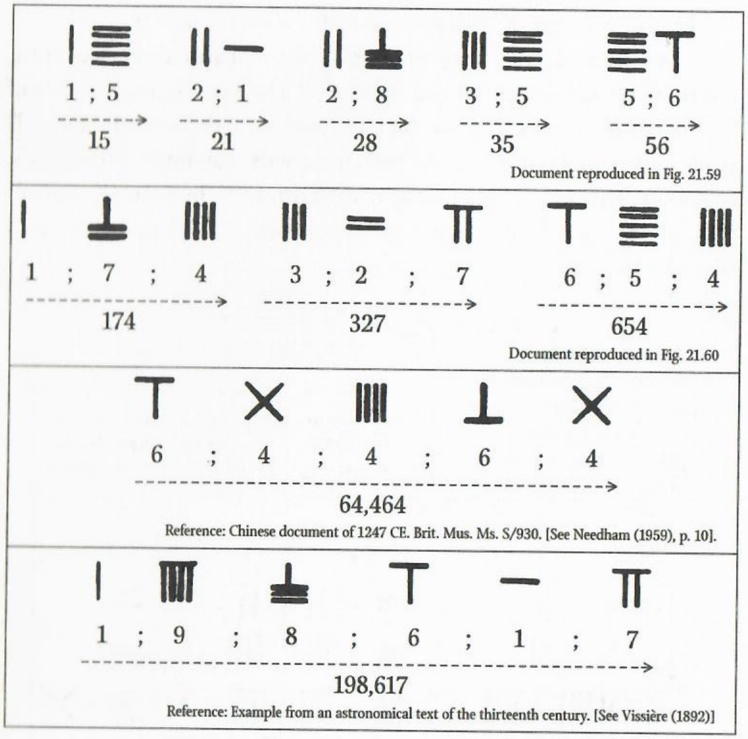

FIG. 21.48. *Examples of numbers written in the Chinese bar notation* (suan zi)

This step was taken at the time of the Han Dynasty (second century BCE to third century CE). This did not solve all the problems there and then, however, since the Chinese mathematicians were to remain unaware of zero for several centuries yet. The following riddle bears witness to this, in the words of the mathematician Mei Wen Ding (1631–1721):

> The character *hai* 亥 has 2 for its head and 6 for its body. Lower the head to the level of the body, and you will find the age of the Old Man of Jiangxian.

In the above, the character playing the main role in the riddle has been written in the *kǎishū* style:

FIG. 21.49. hai

and the riddle remains obscure since the modern character is not the same shape as it was before. According to Chinese sources, however, the riddle dates from long before the Common Era, originating in the middle of the Zhou era (seventh to sixth centuries BCE; see Needham (1959), p. 8). And since at that time Chinese characters were drawn in the *dà zhuàn* ("great seal") style, we must therefore see the character in question drawn in this style if we are to solve the riddle.

In this style, the word was written:

FIG. 21.50. hai

Its "head", therefore, is indeed the figure 2 ═, and its lower part is a "body" consisting of three identical signs each of which resembles the "vertical" symbol for the figure 6 (Fig. 21.47). Arrange the two horizontal lines of the head vertically and on the left-hand side of the body, and you find

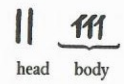

FIG. 21.51.

or, nearly enough,

FIG. 21.52.

The Chinese system being decimal and strictly positional, this represents the number

$$2 \times 1,000 + 6 \times 100 + 6 \times 10 + 6 = 2,666$$

so the solution of the riddle is the number 2,666. But this cannot be an age in years, unless the Old Man of Jiangxian was a Chinese Methuselah. To consider them as 2,666 days would give an absurd answer, since the "Old Man" would then only be seven and a half years old. In fact, this number system had no zero until much later, so the answer can only be one of the numbers 26,660, 266,600, 2,666,000, etc. But since 266,600 or any higher number is out of the question, we are left with 26,660 days. In the riddle, the number sought does not represent days but tens of days: the Old Man of Jiangxian had lived 2,666 tens of days, or about 73 years.

The lack of a sign to represent missing digits also gave rise to confusion. In the first place, a blank space was left where there was no digit, but this was inadequate since numbers like 764, 7,064, 70,640 and 76,400 could easily be confused:

FIG. 21.53.

To avoid such ambiguities, some used signs indicating different powers of 10 from the traditional number-system, so that numbers such as 70,640 and 76,400 would be written as:

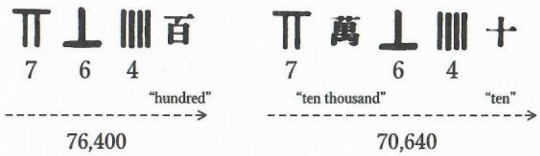

FIG. 21.54.

Others used the traditional expression, therefore writing out in full:

FIG. 21.55.

Yet others placed their numerals in the squares of a grid, leaving an empty square for each missing digit:

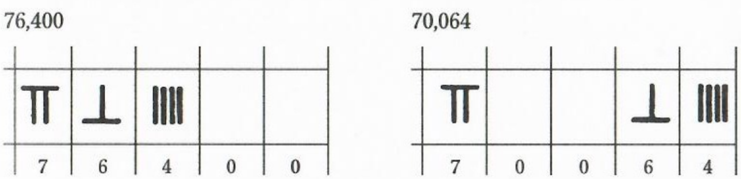

FIG. 21.56.

Only since the eighth century CE did the Chinese begin to introduce a special positional sign (drawn as a small circle) to mark a missing digit (Fig. 21.57); this idea no doubt reached them through the influence of Indian civilisation.

Once this had been achieved, all of the rules of arithmetic and algebra were brought to a degree of perfection similar to ours of the present day.

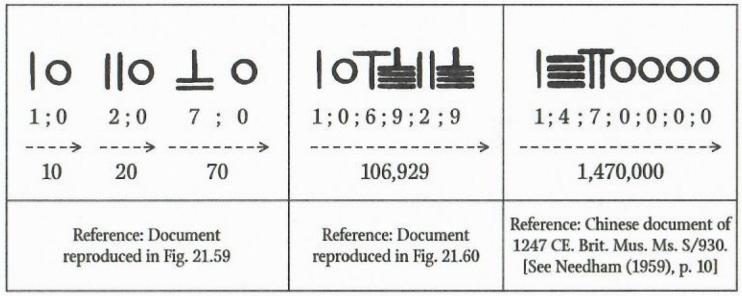

FIG. 21.57. *The use of zero in the Chinese bar numerals*

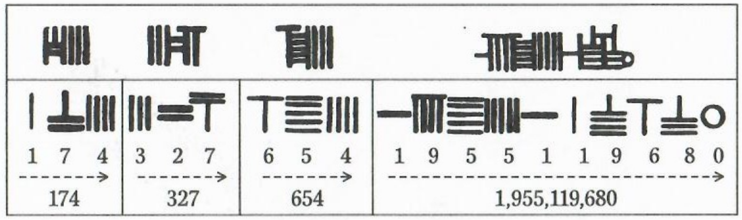

FIG. 21.58. *As a rule, in Chinese manuscripts or printed documents, numbers written in the bar notation are written as monograms, i.e. in a condensed form in which the horizontal strokes are joined to the vertical ones. (Examples taken from the document reproduced in Fig. 21.60)*

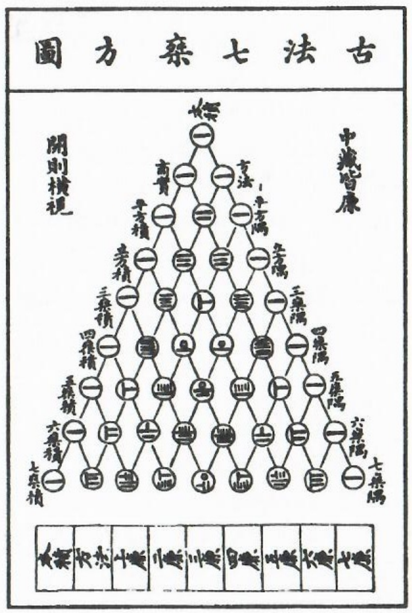

FIG. 21.59A. *Page from a text entitled* Su Yuan Yu Zhian, *published in 1303 by the Chinese mathematician Zhu Shi Jie (see the commentary in the text). [Reproduced from Needham (1959), III, p. 135, Fig. 80]*

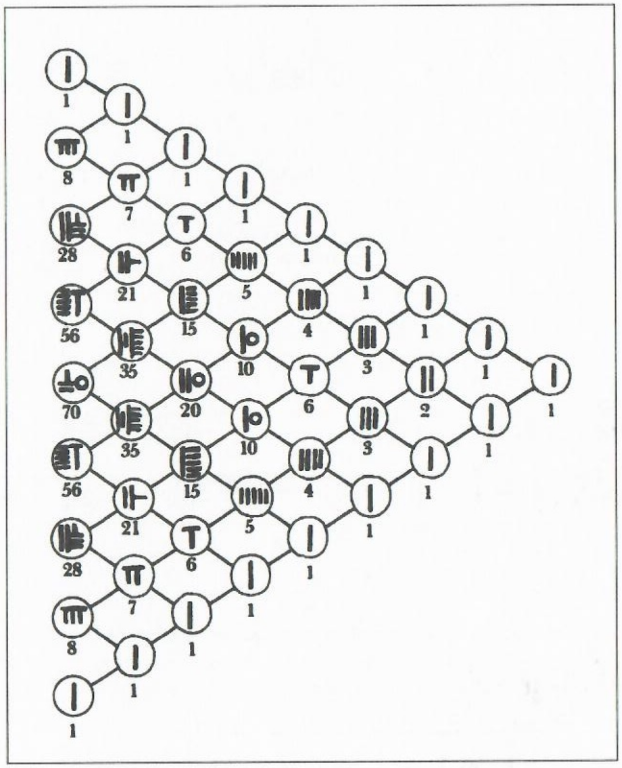

FIG. 21.59B.

Blaise Pascal was long believed in the West to have been the first to discover the famous "Pascal triangle" which gives the numerical coefficients in the expansion of $(a + b)^m$, where m is zero or a positive integer:

BINOMIAL EXPANSIONS	PASCAL'S TRIANGLE
$(a+b)^0 = 1$	1
$(a+b)^1 = a + b$	1 1
$(a+b)^2 = a^2 + 2ab + b^2$	1 2 1
$(a+b)^3 = a^3+3a^2b + 3ab^2+ b^3$	1 3 3 1
$(a+b)^4 = a^4+4a^3b + 6a^2b^2+ 4ab^3+ b^4$	1 4 6 4 1
$(a+b)^5 = a^5+5a^4b + 10a^3b^2+ 10a^2b^3+ 5ab^4+ b^5$	1 5 10 10 5 1
$(a+b)^6 = a^6+6a^5b + 15a^4b^2+ 20a^3b^3+ 15a^2b^4+ 6ab^5+ b^6$	1 6 15 20 15 6 1
- - - - - - - - - - →	- - - - - - - - - - →

In fact, as we can see from Fig. 21.59A, which is schematically redrawn on its side in Fig. 21.59B (to be read from right to left), the Chinese had known of this triangle long before the famous French mathematician.

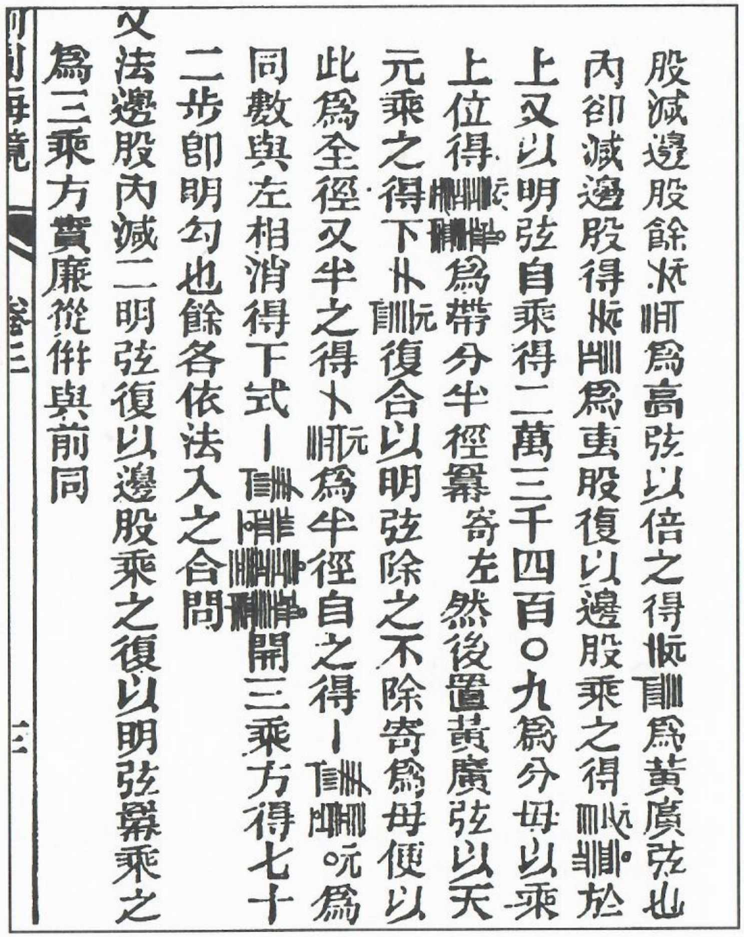
股減邊股餘為高弦以倍之得為黃廣弦也
內卻減邊股得為東股復以邊股乘之得於
上又以明弦自乘得二萬三千四百○九為分母以乘
上位得為帶分半徑冪寄左然後置黃廣弦以天
元乘之得下復合以明弦除之不除寄為母便以
此為全徑又半之得為半徑自之得為
同數與左相消得下式開三乘方得七十
二步即明勾也餘各依法入之合問
又法邊股內減二明弦復以邊股乘之復以明弦冪乘之
為三乘方實廉從併與前同

FIG. 21.60. *Extract from* Ce Yuan Hai Jing, *published in 1248 by the mathematician Li Ye. [Reproduced from Needham (1959), III, page 132, Fig. 79]*

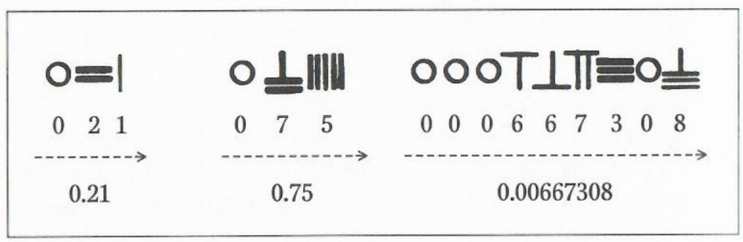

FIG. 21.61. *How Chinese mathematicians extended their positional notation to decimal fractions. Reconstructed examples based on a text from the Mongol period: Biot (1839)*

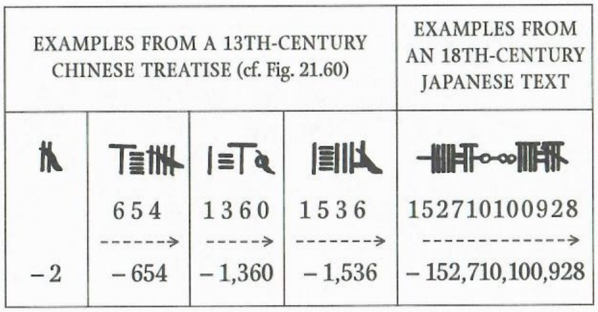

FIG. 21.62A. *Extension of scientific numerical notation to negative numbers. To indicate a negative number, the Chinese and Japanese mathematicians often drew an oblique stroke through the rightmost symbol of the written number. [Menninger (1957); Needham (1959)]*

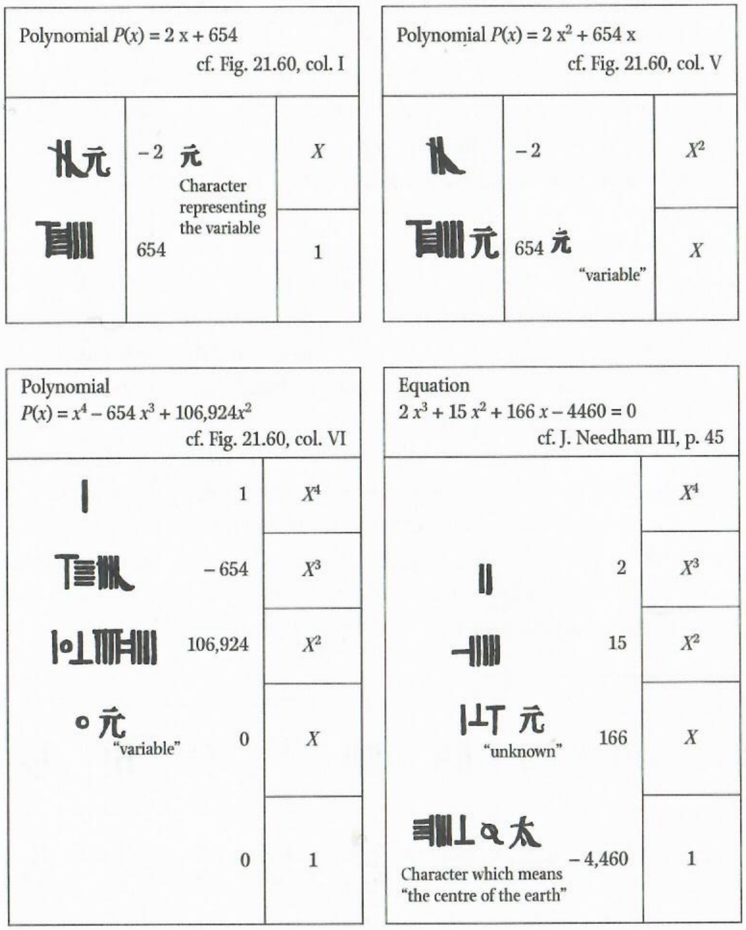

FIG. 21.62B. *Notation for polynomials and for equations in one unknown, used by Li Ye (1178–1265)*

THE CHINESE VERSION OF THE RODS ON THE CHECKERBOARD

Although the numerals discussed above served for writing, they were not used for calculation. For arithmetical calculation, the Chinese used little rods made of ivory or bamboo which were called *chóu* ("calculating rods") which were placed on the squares of a tiled surface or a table ruled like a checkerboard.

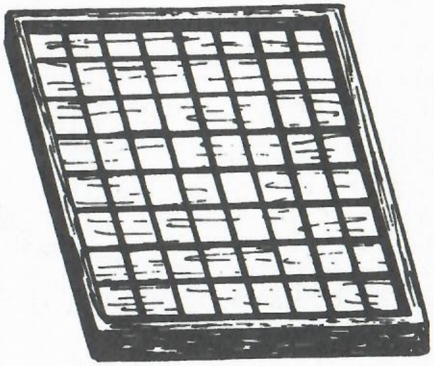

FIG. 21.63. *Model of a Chinese checkerboard used for calculation*

The following story from the ninth century CE is evidence in point. It tells how the Emperor Yang Sun selected his officials for their skill and rapidity in calculation.

> Once two clerks, of the same rank, in the same service, and with the same commendations and criticisms in their records, were candidates for the same position. Unable to decide which one to promote, the superior officer called upon Yang Sun, who had the candidates brought before him and announced: Junior clerks must know how to calculate at speed. Let the two candidates listen to my question. The one who solves it first will have the promotion. Here is the problem:
>
> > A man walking in the woods heard thieves arguing over the division of rolls of cloth which they had stolen. They said that, if each took six rolls there would be five left over; but if each took seven rolls, they would be eight short. How many thieves were there, and how many rolls of cloth?
>
> Yang Sun asked the candidates to perform the calculation with rods upon the tiled floor of the vestibule. After a brief moment, one of the clerks gave the right answer and was given the promotion, and all then departed without complaining about the decision. (See J. Needham in HGS 1, pp. 188–92).

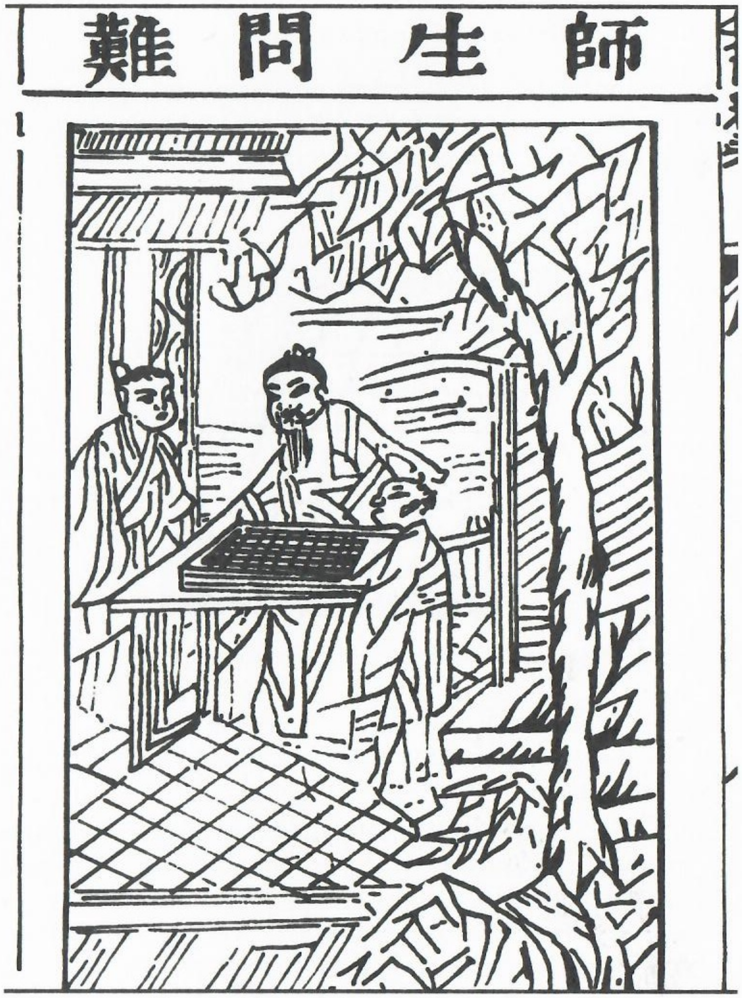

FIG. 21.64. *A Chinese Master teaches the arts of calculation to two young pupils, using an abacus with rods. Reproduced from the* Suan Fa Tong Zong, *published in 1593 in China: [Needham (1959) III, p. 70]*

FIG. 21.65. *An accountant using the arithmetic checkerboard with rods. Reproduced from the Japanese* Shojutsu Sangaka Zue *of Miyake Kenriyû, 1795: (D. E. Smith)*

On an abacus of this kind, each column corresponds to one of the decimal orders of magnitude: from right to left, the first is for the units, the second for the tens, the third for the hundreds, and so on. A given number, therefore, is represented by placing in each column, along a chosen line, a number of rods equal to the multiplicity of the corresponding decimal order of magnitude. For the number 2,645, for example, there would be 5 rods in the first column, 4 in the second, 6 in the third and 2 in the fourth.

For the sake of simplicity, Chinese calculators adopted the following convention (in the words of the old Chinese textbooks of arithmetic): "Let the units lie lengthways and the tens crosswise; let the hundreds be upright and the thousands laid down; let the thousands and the hundreds be face to face, and let the tens of thousands and the hundreds correspond."

The mathematician Mei Wen Ding explains that there was a fear that the different groups might get muddled because there were so many of them. Numbers such as 22 or 33 were therefore represented by two groups of rods, one horizontal and the other vertical, which allowed them to be differentiated. To prevent errors of interpretation, the rods were laid down vertically in the odd-numbered columns (counting from the right), and horizontally in the even-numbered columns (Fig. 21.67).

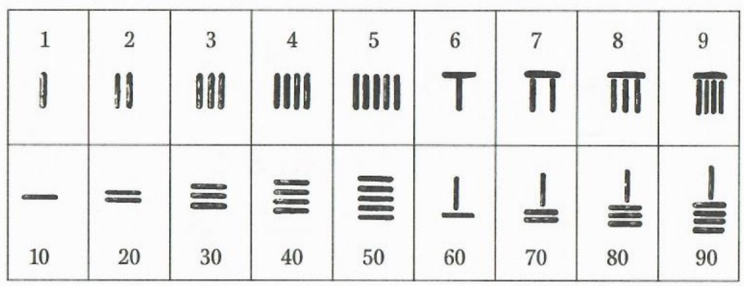

FIG. 21.66. *How the units and tens are represented by rods on the arithmetical checkerboard*