# The Loneliest Numbers

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According to a memorable song from the 1960s, one is the loneliest number, and two can be as bad as one. Maybe so, but the prime numbers have it pretty rough too.

Paolo Giordano explains why in his best-selling novel *The Solitude of Prime Numbers*. It's the melancholy love story of two misfits, two primes, named Mattia and Alice, both scarred by childhood tragedies that left them virtually incapable of connecting with other people, yet who sense in each other a kindred damaged spirit. Giordano writes,

> Prime numbers are divisible only by 1 and by themselves. They hold their place in the infinite series of natural numbers, squashed, like all numbers, between two others, but one step further than the rest. They are suspicious, solitary numbers, which is why Mattia thought they were wonderful. Sometimes he thought that they had ended up in that sequence by mistake, that they'd been trapped, like pearls strung on a necklace. Other times he suspected that they too would have preferred to be like all the others, just ordinary

> numbers, but for some reason they couldn't do it. [. . .]
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> In his first year at university, Mattia had learned that, among prime numbers, there are some that are even more special. Mathematicians call them twin primes: pairs of prime numbers that are close to each other, almost neighbors, but between them there is always an even number that prevents them from truly touching. Numbers like 11 and 13, like 17 and 19, 41 and 43. If you have the patience to go on counting, you discover that these pairs gradually become rarer. You encounter increasingly isolated primes, lost in that silent, measured space made only of ciphers, and you develop a distressing presentiment that the pairs encountered up until that point were accidental, that solitude is the true destiny. Then, just when you're about to surrender, when you no longer have the desire to go on counting, you come across another pair of twins, clutching each other tightly. There is a common conviction among mathematicians that however far you go, there will always be another two, even if no one can say where exactly, until they are discovered.
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> Mattia thought that he and Alice were like that, twin primes, alone and lost, close but not close enough to really touch each other.

Here I'd like to explore some of the beautiful ideas in the passage above, particularly as they relate to the solitude of prime numbers and twin primes. These issues are central to number theory, the subject that concerns itself with the study of whole numbers and their properties and that is often described as the purest part of mathematics.

Before we ascend to where the air is thin, let me dispense with a question that often occurs to practical-minded people: Is number theory good for anything? Yes. Almost in spite of itself, number theory provides the basis for the encryption algorithms used millions of times each day to secure credit card transactions over the Internet and to encode military-strength secret communications. Those algorithms rely on the difficulty of decomposing an enormous number into its prime factors.

But that's not why mathematicians are obsessed with prime numbers. The real reason is that they're fundamental. They're the atoms of arithmetic. Just as the Greek origin of the word "atom" suggests, the primes are "a-tomic," meaning "uncuttable, indivisible." And just as everything is composed of atoms, every number is composed of primes. For example, 60 equals $2 \times 2 \times 3 \times 5$. We say that 60 is a composite number with prime factors of 2 (counted twice), 3, and 5.

And what about 1? Is it prime? No, it isn't, and when you understand why it isn't, you'll begin to appreciate why 1 truly is the loneliest number—even lonelier than the primes.

It doesn't deserve to be left out. Given that 1 is divisible only by 1 and itself, it really should be considered prime, and for many years it was. But modern mathematicians have decided to exclude it, solely for convenience. If 1 were allowed in, it would mess up a theorem that we'd like to be true. In other words, we've rigged the definition of prime numbers to give us the theorem we want.

The desired theorem says that any number can be factored into primes in a *unique* way. But if 1 were considered prime, the uniqueness of prime factorization would fail. For example, 6 would equal $2 \times 3$, but it would also equal $1 \times 2 \times 3$ and $1 \times 1 \times 2 \times 3$ and so on, and these would all have to be accepted as different prime factorizations. Silly, of course, but that's what we'd be stuck with if 1 were allowed in.

This sordid little tale is instructive; it pulls back the curtain on how math is done sometimes. The naive view is that we make our definitions, set them in stone, then deduce whatever theorems happen to follow from them. Not so. That would be much too passive. We're in charge and can alter the definitions as we please—especially if a slight tweak leads to a tidier theorem, as it does here.

Now that 1 has been thrown under the bus, let's look at everyone else, the full-fledged prime numbers. The main thing to know about them is how mysterious they are, how alien and inscrutable. No one has ever found an exact formula for the primes. Unlike real atoms, they don't follow any simple pattern, nothing akin to the periodic table of the elements.

You can already see the warning signs in the first ten primes: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29. Right off the bat, things start badly with 2. It's a freak, a misfit among misfits—the only prime with the embarrassment of being an even number. No wonder "it's the loneliest number since the number one" (as the song says).

Apart from 2, the rest of the primes are all odd . . . but still quirky. Look at the gaps between them. Sometimes they're two spaces apart (like 5 and 7), sometimes four (13 and 17), and sometimes six (23 and 29).

To further underscore how disorderly the primes are, compare them to their straight-arrow cousins the odd numbers: 1, 3, 5, 7, 9, 11, 13, . . . The gaps between odd numbers are always consistent: two spaces, steady as a drumbeat. So they obey a simple formula: the $n$th odd number is $2n - 1$. The primes, by contrast, march to their own drummer, to a rhythm no one else can perceive.