SHERLOCK HOLMES IN BABYLON

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Let me begin by clarifying the title "Sherlock Holmes in Babylon." Lest some members of the Baker Street Irregulars be misled, my topic is the archaeology of mathematics, and my objective is to retrace a small portion of the research of two scholars: Otto Neugebauer, who is a recipient of the Distinguished Service Award, given to him by the Mathematical Association of America in 1979, and his colleague and long-time collaborator, Abraham Sachs. It is also a chance for me to repay both of them a personal debt. I went to Brown University in 1947, and as a new Assistant Professor I was welcomed as a regular visitor to the Seminar in the History of Mathematics and Astronomy. There, with a handful of others, I was privileged to watch experts engaged in the intellectual challenge of reconstructing pieces of a culture from random fragments of the past. (See [4], [5].)

This experience left its mark upon me. While I do not regard myself as a historian in any sense, I have always remained a "friend of the history of mathematics"; and it is in this role that I come to you today.

Let me begin with a sample of the raw materials. Figure 1 is a copy of a cuneiform tablet, measuring perhaps 3 inches by 5. The markings can be made by pressing the end of a cut reed into wet clay. Dating such a tablet is seldom easy. The appearance of this tablet suggests that it may have been made in Akkad in the city of Nippur in the year -1700, about 3,700 years ago.

Confronted with an artifact from an ancient culture, one asks several questions: (i) What is this and what are its properties? (ii) What was its original purpose? (iii) What does this tell me about the culture that produced it? In the History of Science, one expects neither theorems nor rigorous proofs. The subject is replete with conjectures and even speculations; and in place of proof, one often finds mere confirmation: "I believe P implies Q; and because I also believe Q, I therefore also believe P."

In Figure 1, we draw a vertical line to separate the first two columns. In the first column, we recognize what seem to be counting symbols for the numbers from 1 through 9. Paired with these, in the second column we see 9, then 1 and 8, then 2 and 7, and then 3 and 6. This suggests that what we have is a "table of 9's," a multiplication table for the factor 9. Checking further, we see 5 and 4 across from the counting symbol for 6, which confirms the conjecture. However, in the next line we see 7 and then across from it what seems to be a 1 and a 3.

We modify our conjecture; instead of an ordinary decimal system, we are dealing with a hybrid. There is a decimal substratum, using one type of wedge for units and another for tens, but the system is base 60 in the large. The 1 and 3 in fact represent $60+3=63$. We then immediately conjecture that the same wedge symbol will be used for 10, for 60, for $(60)^2$, $(60)^3$, and so on, while the digits will be given in a decimal form.

Thus from a single tablet we might have conjectured a complete sexagesimal numeral system. We would then seek confirmation of this by examining other tablets, hoping to see the same patterns there. Indeed, this was done in the last century, and among the thousands of

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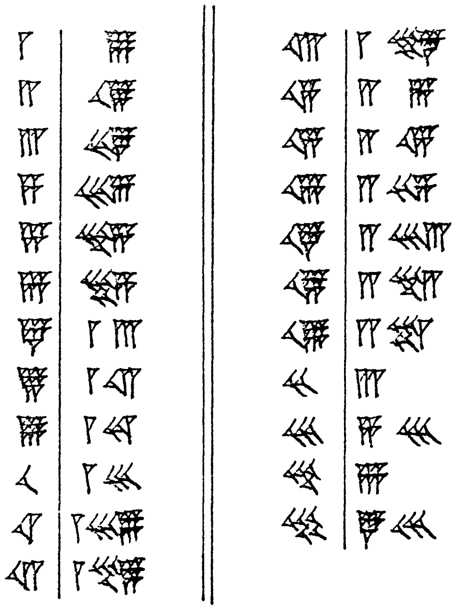


FIG. 1

Babylonian tablets many were found that bear multiplication tables of the same general type as that given in Figure 1, generated by various multiplication factors. There are a great many duplicates.

We find the Babylonian numeral system cumbersome to write. In this paper, base 60 numerals will be written by putting the digits (0 through 59) in ordinary Arabic base ten, and separating consecutive digits by the symbol "/". The "units place" will be on the right as usual. Thus,

$$7/13/28 \quad \text{represents} \quad 28+13(60)+7(60)^2=26{,}008$$

Addition is easy:

$$\begin{array}{r} 14/28/31 \\ 3/35/45 \\ \hline 18/4/16 \end{array}$$

If the tablets that bear multiplication tables are catalogued, something strange is seen. Many tables of 9's, 12's, etc., are found; but there are also multiplication tables for unlikely factors, while many tables we would have expected never appear. In Figure 2, we list those that occur frequently.

We are left with three puzzles: (i) Why are some tables missing? (For example, 7, 11, 13, 14, etc.?) (ii) Why are there tables with factors such as 3/45, 7/12, 7/30, and 44/26/40? (iii) Why are there so many tablets with exactly the same multiplication tables on them? Some clues are found; for example, there are tablets that contain two versions of the same multiplication table, one done neatly and one less neatly and perhaps with an error or two. I am sure that a familiar picture comes immediately to your mind: a cluster of students, all engaged in copying a model table provided by the teacher who will shortly be grading their efforts. Are we not correct to infer that in Nippur there was probably an extensive school for scribes who were in training to become bureaucrats or priests?

To help answer the first two questions, let us examine another tablet, which for convenience I have transcribed into the slash notation. (See Fig. 3.) This again fits the pattern of two matched

Factors Used for
Multiplication Tables

2	18	1/15=75	7/12=432
3	20	1/20=80	7/30=450
4	24	1/30=90	8/20=500
5	25	1/40=100	12/30=750
6	30	2/15=135	16/40=1000
8	36	2/24=144	22/30=1350
9	40	2/30=150	44/26/40=160,000
10	45	3/20=200	
12	48	3/45=225	and a scattering of others
15	50	4/30=270	
16		6/40=400	

FIG. 2

2	30	16	3/45	45	1/20
3	20	18	3/20	48	1/15
4	15	20	3	50	1/12
5	12	24	2/30	54	1/6/40
6	10	25	2/24	1/4	56/15
8	7/30	27	2/13/20	1/12	50
9	6/40	30	2	1/15	48
10	6	32	1/52/30	1/20	45
12	5	36	1/40	1/21	44/26/40
15	4	40	1/30		

FIG. 3

columns, and we look for an explanation. We note at once that in the first few rows the product of the adjacent column numbers is always 60. There seem to be some exceptions, however. With the pair 9 and 6/40, this product is

$$(9)\times(6/40)=(9)\times(400)=3600$$

and again

$$(16)\times(3/45)=(16)\times(225)=3600$$

while still further down, we see

$$(27)\times(2/13/20)=(27)\times(8000)=216{,}000.$$

The solution becomes obvious if we write these products in Babylonian form; since 60 is 1/0, 3600 is 1/0/0, and 216,000 is 1/0/0/0. For confirmation, look at the last entry in the table:

$$(1/21)\times(44/26/40)=(81)\times(160{,}000)=12{,}960{,}000$$
$$=1/0/0/0/0.$$

If we now follow the Babylonian practice of omitting terminal zeros, we see that Figure 3 is merely a table of reciprocals, written in "sexagesimal floating point." If A is an integer in the first column, the integer paired with it in the second column, A^R, is one chosen so that their product would be written as "1," meaning *any* suitable power of 60. The integers that appear in the table will always be factorable into powers of 2, 3, and 5, since these have terminating reciprocals in base 60. The term "floating-point arithmetic" is today a computer concept but is also understandable to anyone who has used a slide rule or worked with logarithms; the concept would also have been familiar to medieval astronomers who multiplied large numbers by the device called "posthaphaeresis."

Now that Figure 3 is understood, we can answer the two puzzles left hanging on the previous page. Observe that the integers used to generate multiplication tables, as seen in Figure 2, mostly come from the standard reciprocal table. (There are also tablets that contain nonstandard reciprocals, reciprocals of such numbers as 7, 11, etc., of necessity given in terminating approximate form.) In floating point, $B \div A = B \times A^R$. Thus the combination of a set of multiplication tables and a reciprocal table makes it easy to carry out floating-point division, provided that the divisor is one of the "nice" numbers in base 60, of the form $2^\alpha 3^\beta 5^\gamma$. For example, let us divide 417 by 24; in base 60, this will be $6/57 \div 24 = 17/22/30$.

Method: $6/57 \div 24 = (6/57) \times (24)^R = (6/57) \times (2/30)$:

$$\begin{aligned} 6/57 \times 2 &= 12 + 1/54 = 13/54 \\ 6/57 \times 30 &= 3 + 28/30 = \underline{3/28/30} \\ \text{answer} &\qquad\;\; = 17/22/30 \end{aligned}$$

The last steps in this calculation are easier if one recalls that $30 = 2^R$, so that multiplication by 30 is the same as halving. (Of course the scribe must be sure to keep track of the actual magnitudes and place values.)

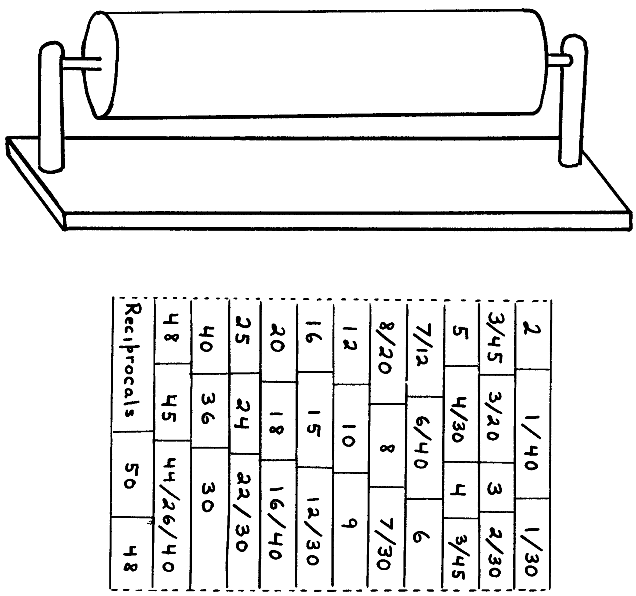


FIG. 4

That common calculations were made in this fashion becomes even more plausible in the light of one remarkable discovery. This is an inscribed cylinder, carrying on its curved face a copy of the standard reciprocal table and each of the standard multiplication tables. (In Figure 4, we show this restored, with each multiplication table indicated by its generator.) With the help of this cylinder, perhaps mounted on a stand, a scribe could easily keep track of taxes and calculate wages; perhaps we have here the Babylonian version of a slide rule or desk calculator!

With this brief introduction to the arithmetic of the Babylonians, we turn to another tablet whose mathematical nature had been overlooked until the work of Neugebauer and Sachs. It is in the George A. Plimpton Collection, Rare Book and Manuscript Library, at Columbia University, and usually called Plimpton 322. (See Fig. 5, which is reproduced here by permission of the Library.) The left side of this tablet has some erosion; traces of modern glue on the left edge suggest that a portion that had originally been attached there has since been lost or stolen. Since it was bought in a marketplace, one may only conjecture about its true origin and date,

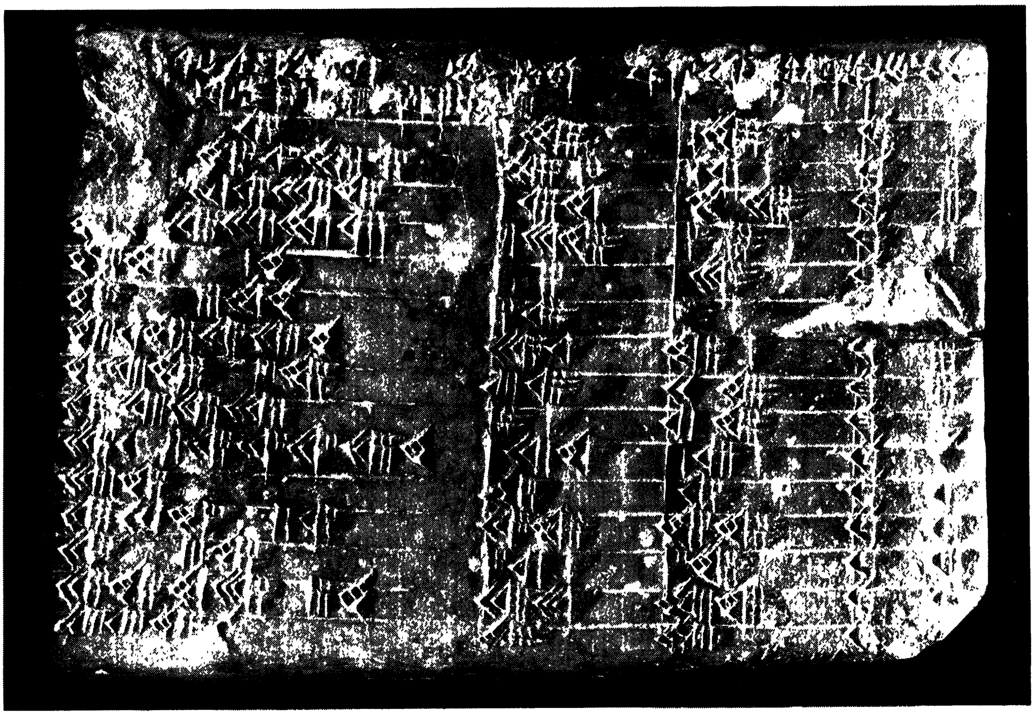


FIG. 5. Plimpton 322

Plimpton 322

Column A	Column B	Column C
15	1/59	2/49
58/14/50/6/15	56/7	3/12/1
1/15/33/45	1/16/41	1/50/49
5 29/32/52/16	3/31/49	5/9/1
48/54/ 1/40	1/5	1/37
47/ 6/41/40	5/19	8/1
43/11/56/28/26/40	38/11	59/1
41/33/59/ 3/45	13/19	20/49
38/33/36/36	9/1	12/49
35/10/2/28/27/24/26/40	1/22/41	2/16/1
33/45	45	1/15
29/21/54/ 2/15	27/9	48/49
27/ 3/45	7/12/1	4/49
25/48/51/35/6/40	29/31	53/49
23/13/46/40	56	53

FIG. 6

although the style suggests about −1600 for the latter. As with most such tablets, this had been assumed to be a commercial account or inventory report. We will attempt to show why one can be led to believe otherwise.

First, let us transcribe it into the slash notation, as seen in Figure 6. We have reproduced the three main columns, which we have labeled A, B, and C. We note that there are gaps in column A, due to the erosion. However, it seems apparent that the numbers there are steadily decreasing. We note that some of the numerals there are short and some long, apparently at random. In contrast with this, all the numerals in columns B and C are rather short, and we do not see any evidence of general monotonicity.

B	C
119	169
3367	11521
4601	6649
12709	18541
65	97
319	481
2291	3541
799	1249
541	769
4961	8161
45	75
1679	2929
25921	289
1771	3229
56	53

FIG. 7

$C+B$	$C-B$
288	50
14888	8154
11250	2048
31250	5832
162	32
800	162
5832	1250
2048	450
1310	228
13132	3200
120	30
4608	1250
26210	−25632
5000	1458
109	−3

FIG. 8

Since it is easier for us to work with Arabic numerals, let us translate columns B and C into these numerals and look for patterns. (See Fig. 7.) We see at once that B is smaller than C, with only two exceptions. Also, playing with these numbers, we find that column B contains exactly one prime, namely, 541, while column C contains eight numbers that are prime.

In the first 20,000 integers, there are about 2,300 primes, which is about 10 percent; among 15 integers, selected at random from this interval, we might, then, expect to see one or two primes, but certainly not eight! This at once tells us that the tablet is mathematical and not merely

B	C	(a,b)
119	169	12, 5
3367	11521	?
4601	6649	75, 32
12709	18541	125, 54
65	97	9, 4
319	481	20, 9
2291	3541	54, 25
799	1249	32, 15
541	769	?
4961	8161	81, 40
45	75	?
1679	2929	48, 25
25921	289	?
1771	3229	50, 27
56	53	?

FIG. 9

Corrected Version

B	C	(a,b)
119	169	12, 5
3367	4825	64, 27
4601	6649	75, 32
12709	18541	125, 54
65	97	9, 4
319	481	20, 9
2291	3541	54, 25
799	1249	32, 15
481	769	25, 12
4961	8161	81, 40
45	75	1, $\frac{1}{2}$=30
1679	2929	48, 25
161	289	15, 8
1771	3229	50, 27
56	106	9, 5

FIG. 10

arithmetical. (Imagine your feelings if you were to find a Babylonian tablet with a list of the orders of the first few sporadic simple groups.)

Encouraged, one attempts to find further visible patterns, for example, by combining the entries in columns B and C in various ways. One of the earliest tries is immediately successful. In Figure 8, we show the results of calculating $C+B$ and $C-B$. If you are sensitive to arithmetic you will note that, in almost every case, the numbers are each twice a perfect square.

If $C+B=2a^2$ and $C-B=2b^2$, then $B=a^2-b^2$ and $C=a^2+b^2$. Thus the entries in these columns could have been generated from integer pairs (a,b). In passing, we note that b, being $(a-b)(a+b)$, is not apt to be prime; on the other hand, when a and b are relatively prime, every prime of the form $4N+1$ can be expressed as a^2+b^2.

In Figure 9, we have recopied columns B and C, together with the appropriate pairs (a,b) in the cases where this representation is possible. As a further confirmation that we are on the right track, we note that in every such pair the numbers a and b are both "nice," that is, factorable in terms of 2, 3, and 5. In five cases, the pattern breaks down and no pair exists. It will be a further confirmation if we can explain these discrepancies as errors made by the scribe who produced the tablet. We make a simple hypothesis and assume that B and C were each computed independently from the pair (a,b) and that a few errors were made but each affected only one number in each row. Thus in each vacant place we will assume that either B or C is correct and the other wrong, and attempt to restore the correct entry. Since we do not know the correct pair (a,b) we must find it; because of the evidence in the rest of the table, we insist that an acceptable pair must be composed of "nice" sexagesimals.

We start with line 9; here, $B=541$, which happens to be the only prime in Column B. We therefore assume B is wrong and C is correct, and thus write $C=769=a^2+b^2$. This has a single solution, the pair (25,12). (We also note that both happen to be nice sexagesimals.) If this is correct, then B should have been $(25)^2-(12)^2=481$, instead of 541 as given. Is there an obvious explanation for this mistake? Yes, for in slash notation, $541=9/1$ and $481=8/1$. The anomaly in line 9 seems to be merely a copy error.

Turn now to line 13; here, B is far larger than C, which is contrary to the pattern. Assume that B is in error and C is correct, and again try $C=289=a^2+b^2$. There is a "nice" unique solution, (15,8), and using these, we are led to conjecture that the correct value of B is $(15)^2-(8)^2=161$. Again, we ask if there is an obvious explanation for arriving at the incorrect value given, 25921. A partial answer is immediate: $(161)^2=25921$; so that for some reason the scribe recorded the *square* of the correct value for B.

Continuing, consider line 15. Since $B=56$ and $C=53$, we have $B>C$, which does not match the general pattern. However, it is not clear whether B is too large or C too small. Trying the first, we assume C is correct and solve $53=a^2+b^2$, obtaining the unique answer (7,2). We reject this, since 7 is not a nice sexagesimal. Now assume that B is correct, and write $56=a^2-b^2=(a+b)(a-b)$. This has two solutions, (15,13) and (9,5). We reject the first and use the second, obtaining $9^2+5^2=106$ as the correct value of C. Seeking an explanation, we note that the value given by the scribe, 53, is exactly *half* of the correct value.

Turning now to line 2 of Figure 9, we have $B=3367$ and $C=11521$, either of which might be correct. Assume that $C=a^2+b^2$ and find two solutions (100,39) and (89,60). While 100 and 60 are nice, 39 and 89 are not, so we reject both pairs and assume that B is correct. Writing $3367=(a-b)(a+b)$ and factoring 3367 in all ways, we find four pairs: (1684,1683), (244,237), (136,123), (64,27), of which we can accept only the last. This yields $(64)^2+(27)^2=4825$ as the correct C. Comparing this with the number 11521 that appeared on the tablet, we see no immediate naive explanation for the error. For example, since $4825=1/20/25$ and $11521=3/12/1$, it does not seem to be a copy error. Without an explanation, we may have a little less confidence in this reconstruction of the entries in line 2.

The last misfit in the table is line 11, where we have $B=45$ and $C=75$. This is unusual also because this is the only case where B and C have a common factor. The sums-and-differences-of-squares pattern failed because neither $C+B=120$ nor $C-B=30$ is twice a square. However, everything becomes clearer if we go back to base 60 notation and remember that we use floating point; for $120=2/0$, which is twice $1/0$ and which we can also write as 1, clearly a perfect square. In the same way, 30 is twice 15, which is also 4^R and which is the square of 2^R. The pattern is preserved and no corrections need be made in the entries: with $a=1=1/0$ and $b=\frac{1}{2}=2^R=30=0/30$, we have $a^2=1/0$ and $b^2=0/15$, and

$$C=a^2+b^2=1/0+0/15=1/15=75$$
$$B=a^2-b^2=1/0-0/15=0/45=45.$$

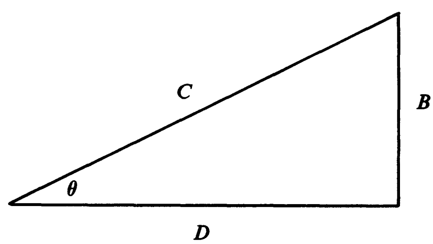
(Another aspect of the line 11 entries will appear later.)

With this, we have completed the work of editing the original tablet. In Figure 10, we give a corrected table for columns B and C, together with the appropriate pairs (a,b) from which they can be calculated.

It is now the time to raise the second canonical question: What was the purpose behind this tablet? Speculation in this direction is less restricted, since the road is not as well marked. We can begin by asking if numbers of the form a^2-b^2 and a^2+b^2 have any special properties. In doing so, we run the risk of looking at ancient Babylonia from the twentieth century, rather than trying to adopt an autochthonous viewpoint. Nevertheless, one relation is extremely suggestive, involving both algebra and geometry. For any numbers (integers) a and b,

$$(a^2-b^2)^2+(2ab)^2=(a^2+b^2)^2 \qquad (*)$$

In addition, if we introduce $D=2ab$, then B, C, and D can form a right-angled triangle with $B^2+D^2=C^2$. And finally, these formulas generate *all* Pythagorean triplets (triangles) from the integer parameters (a,b). (See Fig. 11.)



$B=a^2-b^2,\quad D=2ab,\quad C=a^2+b^2$

FIG. 11

There is no independent information showing that these facts were known to the Babylonians at the time we conjecture that this tablet was inscribed, although, as will appear later, their algebra had already mastered the solution of quadratic equations. If the tablet indeed is connected with this observation, then the unknown column A numbers ought to be connected in some way with the same triangle. The next step is, then, to proceed as before and try many different combinations of B, C, and D, in hopes that one of these will approximate the entries in column A. Slopes and ratios are an obvious starting point, so one calculates $C \div B, C \div D, B \div D$, etc. After discarding many failures, one arrives at the combination $(B \div D)^2$. In Figure 12, we give the values of this expression, calculated from the corrected values of B and using the hypothetical values of (a,b) to find D. (We remark that it was very helpful to have a programmable pocket calculator that could be trained to work in sexagesimal arithmetic!)

If we now return to Figure 6 and compare the numerals given there in column A with those that appear in Figure 12, we see that there is almost total agreement. For example, in line 10 we have exact duplication of an eight-digit sexagesimal! On probabilistic grounds alone, this is an overwhelming confirmation. Of course, at the top of the tablet where there were gaps due to erosion, Figures 6 and 12 are not the same, but it is evident that the calculated data in Figure 12 can be regarded as filling in the gaps. There are two minor disagreements in the two tables. In line 13, the tablet does not show an internal "0" that is present in Figure 12. This could have been the custom of the scribe in dealing with such an event. In line 8, the scribe has written a digit "59" where there should have been a consecutive pair of digits, "45/14". Since $59=45+14$, it is not difficult to invent several different ways in which an error of this sort could have been made.

Calculated Values of $(B+D)^2$

line	1	59/0/15
	2	56/56/58/14/50/6/15
	3	55/7/41/15/33/45
	4	53/10/29/32/52/16
	5	48/54/1/40
	6	47/6/41/40
	7	43/11/56/28/26/40
	8	41/33/45/14/3/45
	9	38/33/36/36
	10	35/10/2/28/27/24/26/40
	11	33/45
	12	29/21/54/2/15
	13	27/0/3/45
	14	25/48/51/35/6/40
	15	23/13/46/40

FIG. 12

It should be remarked that Neugebauer and Sachs did not use $(B+D)^2$ as a source for column A but rather $(C+D)^2$. Because of the relationship between B and C, and formula (*), one sees that $(C+D)^2=(B+D)^2+1$. Thus, the only effect of the change would be to introduce an initial "1/" before all the sexagesimals that appear in Figure 12, and the reason for their choice was that they believed that this was true for column A on the Plimpton tablet. Others who have examined the tablet do not agree. (I have not seen the tablet, and I do not believe it matters which alternative is used.)

We now know the relationship of columns A, B, and C. Referring to Figure 11, C is the hypotenuse, B the vertical side, and A is the square of the slope of the triangle; thus, in modern notation $A=\tan^2\theta$. It is interesting to observe that the anomalous case of line 11, with $B=45$ and $C=75$, turns out to be the familiar 3,4,5 triangle; in the Babylonian case, this would seem to have been the $\frac{3}{4},1,\frac{5}{4}$ triangle, since $45=3\times 4^R$ and $75=1/15=5\times 4^R$. Of course the triangle, the side D, and the parameters (a,b) are all constructs of ours and not immediately visible in the original tablet. All that we can assert without controversy is that $A=B^2\div(C^2-B^2)$.

Let us reexamine some of our reasoning. In lines 2, 9, 13, and 15, the scribe recorded correct values for A but incorrect values for C, B, B, and C, respectively. This suggests strongly that A was not calculated directly from the values of B and C, but that A, B, and C were all calculated independently from data that do not appear on the tablet; our hypothetical pair (a,b) gains life. (Of course there is the possibility that the tablet before us is merely a copy from another master tablet.) In either case, it seems odd that column A should be error free while columns B and C, involving simpler numbers, should have four errors.

Other questions can be raised. If, as argued by Neugebauer, the purpose of the tablet was to record a collection of integral-sided Pythagorean triangles (triplets), why do we not see the values of D, or at least the useful parameters (a,b)? And why would one want the values in column A which are squares of the slope? And why should the entries be arranged in an order that makes the numbers A decrease monotonically?

Variants of this explanation have been proposed. If one computes the values of the angle θ for each line of the tablet, they are seen to decrease steadily from about 45° to about 30°, in steps of about 1°. Is this an accident? Could this tablet be a primitive trigonometric table, intended for engineering or astronomic use? But again, why is $\tan^2\theta$ useful [3],[6]?

Additional confirmation of such a hypothesis could be given by an outline of a computation procedure leading to the tablet, which makes all of the errors plausible and also shows why they would have occurred preferentially in columns B and C. (See [1],[4],[7].)

Building upon an earlier suggestion of Bruins, an intriguing explanation has been recently proposed by Voils. In Nippur, a large number of "school texts" have been found, many containing arithmetic exercises. Among these, a standard puzzle problem is quite common. The student is given the difference (or sum) of an unknown number and its reciprocal and asked to find the number. If x is the number (called "igi") and x^R is its reciprocal (called "igibi"), then the student is to solve the equation $x - x^R = d$. Thus, the "igi and igibi" problems are quadratic equations of a standard variety.

The school texts teach a specific solution algorithm: "Find half of d, square it, add 1, take the square root, and then add and subtract half of d." This is easily seen to be nothing more than a version of the quadratic formula, tailored to the "igi and igibi" problems. Voils connects this class of problems, and the algorithm above, with the Plimpton tablet as follows.

First, assume with Bruins that the tablet was computed not from the pair (a,b) but from a single parameter, the number $x = a \div b$. Since a and b are both "nice," the number x and its reciprocal x^R can each be calculated easily. Indeed, $x = a \times b^R$ and $x^R = b \times a^R$, and a^R and b^R, each appear in a standard reciprocal table. Next observe that

$$B = a^2 - b^2 = (ab)(x - x^R)$$
$$C = a^2 + b^2 = (ab)(x + x^R)$$
$$A = \left(\frac{B}{D}\right)^2 = \left\{\frac{1}{2}(x - x^R)\right\}^2.$$

This shows that the entries A, B, C in the Plimpton tablet could have been easily calculated from a special reciprocal table that listed the paired values x and x^R. Indeed, the numbers B and C can be obtained from $x \pm x^R$ merely by multiplying these by integers chosen to simplify the result and shorten the digit representation. (See [1],[2],[7].)

Voils adds to this suggestion of Bruins the observation that the numbers A are exactly the results obtained at the end of the second step in the solution algorithm, $(d/2)^2$, applied to an igi-igibi problem whose solution is x and x^R. Furthermore, the numbers B and C can be used to produce other problems of the same type but having the same intermediate results in the solution algorithm. Thus Voils proposes that the Plimpton tablet has nothing to do with Pythagorean triplets or trigonometry but, instead, is a pedagogical tool intended to help a mathematics teacher of the period make up a large number of igi-igibi quadratic equation exercises having known solutions and intermediate solution steps that are easily checked [7].

It is possible to point to another weak confirmation of this last approach. Suppose that we want a graduated table of numbers x and their reciprocals x^R. We start with the class of *all* pairs (a,b) of relatively prime integers such that $b < a < 100$ and each integer a and b is "nice," factorable into powers of 2, 3, and 5. It is then easy to find the terminating Babylonian representation for both $x = a \div b$ and for $x^R = b \div a$. Make a table of these, arranged with x decreasing. Impose one further restriction:

$$\sqrt{3} < x < 1 + \sqrt{2}\ .$$

(This corresponds to the limitation $30° < \theta < 45°$, where θ is the base angle in the triangle in Figure 11.)

Then, the resulting list of pairs will coincide with that given in Figure 10, the corrected

Plimpton table, except for three minor points. The pair (16,9) does not appear, the pair (125,54) does appear, and instead of the pair (2,1) we have the pair $(1,\frac{1}{2})$; in passing, we recall that the last pair yields the standard 3,4,5 Pythagorean triangle.

Unlike Doyle's stories, this has no final resolution. Any of these reconstructions, if correct, throws light upon the degree of sophistication of the Babylonian mathematician and breathes life into what was otherwise dull arithmetic. For other vistas into the past, especially those that show us the beginnings of computational astronomy, I refer the reader to the bibliography. I can do no better than to close with an analogy used by Neugebauer:

> In the "Cloisters" of the Metropolitan Museum in New York there hangs a magnificent tapestry which tells the tale of the Unicorn. At the end we see the miraculous animal captured, gracefully resigned to his fate, standing in an enclosure surrounded by a neat little fence. This picture may serve as a simile for what we have attempted here. We have artfully erected from small bits of evidence the fence inside which we hope to have enclosed what may appear as a possible living creature. Reality, however, may be vastly different from the product of our imagination; perhaps it is vain to hope for anything more than a picture which is pleasing to the constructive mind, when we try to restore the past.
>
> —*The Exact Sciences in Antiquity* (p. 177)

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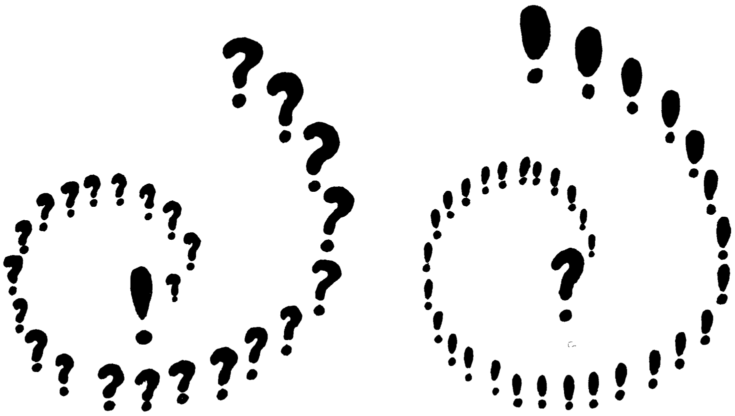
35.

OPEN AND CLOSED APPROACHES TO UNDERSTANDING



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