astronomical reckonings right down to the dawn of the Common Era, the "learned" numerals of Babylon were a direct inheritance of Sumer, whose memory they have perpetuated, directly and indirectly, right down to the present day.

THE POSITIONAL SEXAGESIMAL SYSTEM OF THE LEARNED MEN OF MESOPOTAMIA

Although we cannot be sure about the exact date, the first real idea of a positional numeral system arose amongst the mathematicians and astronomers of Babylon in or around the nineteenth century BCE.

The Mesopotamian scholars' abstract numerals were derived from the ancient Sumerians' sexagesimal figures, but constituted a system far superior to anything else in the ancient world, anticipating modern notation in all respects save for the different base and the actual shapes used for the numerals.

Unlike the "ordinary" Assyro-Babylonian notation used for everyday business needs, the learned system used base 60 and was strictly positional. Thus a group of figures such as

[3; 1; 2]

which in modern decimal positional notation would express:

$$3 \times 10^2 + 1 \times 10 + 2$$

signified to Babylonian mathematicians and astronomers:

$$3 \times 60^2 + 1 \times 60 + 2$$

Similarly, the sequence [1; 1; 1; 1] which in our system would mean $1 \times 10^3 + 1 \times 10^2 + 1 \times 10 + 1$ (or 1,000 + 100 + 10 + 1) signified in the Babylonian system $1 \times 60^3 + 1 \times 60^2 + 1 \times 60 + 1$ (or 216,000 + 3,600 + 60 +1).

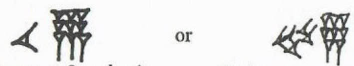
Instances of this system of numerals have been known since the very dawn of Assyriology, in the mid-nineteenth century, and, thanks to excavations made throughout Mesopotamia and Iraq at that time, many examples have come to rest in the great European museums (Louvre, British Museum, Berlin) and in the university collections at Yale, Columbia, Pennsylvania, etc. The types of document in which the learned system is used (and which come from Elam and Mari, as well as from Nineveh, Larsa, and other Mesopotamian cities) are for the most part as follows: tables intended to assist numerical calculation (e.g. multiplication tables, division tables, reciprocals, squares, square roots, cubes, cube roots, etc.); astronomical tables; collections of practical arithmetical and elementary geometrical exercises; lists of more or less complex mathematical problems.

The system is sexagesimal, which is to say that 60 units of one order of magnitude constitute one unit of the next (higher) order of magnitude. The numbers 1 to 59 constitute the units of the first order, multiples of 60 constitute the second order, multiples of 3,600 (sixty sixties) constitute the third order, multiples of 216,000 (the cube of 60) constitute the fourth order, and so on.

In fact, there were really only two signs in the system: a vertical wedge representing a unit, and a chevron representing 10:

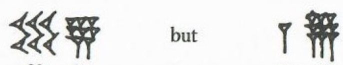
1 10

Numbers from 1 to 59 inclusive were built on the principle of addition, by an appropriate number of repetitions of the two signs. Thus the numbers 19 and 58 were written



(1 chevron + 9 wedges) or (5 chevrons + 8 wedges)

So far the system is exactly the same as its predecessors. However, beyond 60, the learned system became strictly positional. The number 69, for instance, was not written



60 9 but [1; 9]

For example, this is how Asarhaddon, king of Assyria from 680 to 669 BCE, justified his decision to rebuild Babylon (wrecked by his father Sennacherib in 689 BCE) rather sooner than the holy writ prescribed:

> After inscribing the number 70 for the years of Babylon's desertion on the Tablet of Fate, the God Marduk, in his pity, changed his mind. He turned the figures round and thus resolved that the city would be reoccupied after only eleven years. [From *The Black Stone*, trans. J. Nougayrol]

The anecdote takes on its full meaning only in the light of Babylonian sexagesimal numbering. To begin with, Marduk, chief amongst the gods in the Babylonian pantheon, decides that the city will remain uninhabited for 70 years, and, to give full force to his decision, inscribes on the Tablet of Fate the signs:

FIG. 13.40A. [1; 10] ([1; 10] = $1 \times 60 + 10$)

Thereafter, feeling compassion for the Babylonians, Marduk inverts the order of the signs in the expression, thus:

10 . 1 (= 10 + 1)

FIG. 13.40B.

Since the new expression represents the number 11, Marduk decreed that the city would remain uninhabited only for that length of time, and could be rebuilt thereafter. The anecdote shows that the Mesopotamian public in general was at least aware of the rule of position as applied to base 60.

In the Babylonian system, therefore, the value of a sign varied according to its position in a numerical expression. The figure for 1 could for instance express

- a unit in first position from the right,
- a sixty in the second position,
- sixty sixties or 60^2 in third position,

and so on.

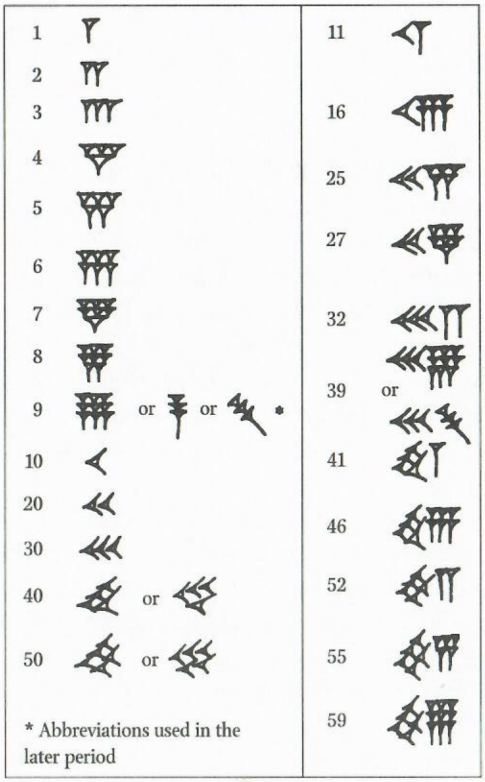


FIG. 13.41. *Representations of the fifty-nine significant units of the learned Mesopotamian numeral system*

For instance, to write the number 75 (one sixty and fifteen units) you put a "15" in first position and a "1" in second position, thus:

[1 ; 15] (= 1 × 60 + 15 = 75)

FIG. 13.42.

And to write 1,000 (16 sixties and 40 units) you put a "40" in first position and a "16" in second position, thus:

[16 ; 40] (= 16 × 60 + 40 = 1,000)

FIG. 13.43.

Conversely, an expression such as

[48 ; 20 ; 12]

FIG. 13.44.

expresses the number:

$$48 \times 60^2 + 20 \times 60 + 12 = 48 \times 3{,}600 + 20 \times 60 + 12 = 174{,}012$$

in exactly the same way as we would express "174,012 seconds" as:

48 h 20m 12s

Similarly, an expression such as

[1 ; 50 + 7 ; 30 + 6 ; 10 + 5] or [1 ; 57 ; 36 ; 15]

FIG. 13.45.

symbolises, in the minds of the Babylonian scholars, the number:

$$1 \times 60^3 + 57 \times 60^2 + 36 \times 60 + 15 \ (= 423{,}375)$$

The next examples come from one of the most ancient Babylonian mathematical tablets known (British Museum, BM 13901, dating from the period of the first kings of the Babylonian Dynasty), a collection of problems relating the solution of the equation of the second degree:

[17 ; 46 ; 40]
(= 17 × 60² + 46 × 60 + 40)
64,000

FIG. 13.46.

[1 ; 57 ; 46 ; 40]
(= 1 × 60³ + 57 × 60² + 46 × 60 + 40)
424,000

FIG. 13.47.

The difference between Sumerian numbers and the Babylonian "learned" system was simply this: the Sumerians relied on addition, the Babylonians on the rule of position. This can easily be seen by comparing the Sumerian and Babylonian expressions for the two numbers 1,859 and 4,818:

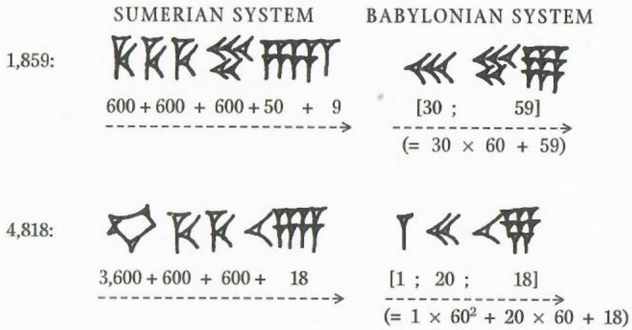


FIG. 13.48A. FIG 13.48B.

THE TRANSITION FROM SUMERIAN TO LEARNED BABYLONIAN NUMERALS

One of the reasons for the "invention" of the learned Babylonian system is easy to understand – it was the "accident" which gave 1 and 60 the same written sign in Sumerian, and which originally constituted the main difficulty of using Sumerian numerals for arithmetical operations.

Moreover, the path to the discovery of positionality had been laid out in the very earliest traces of Sumerian civilisation. The two basic units were represented, first of all, by the same name, *geš* (see Fig. 8.5A and 8.5B above); then, in the second half of the fourth millennium BCE, they were represented by objects of the same shape (the small and large cone) (see Fig. 10.4 above); then, from 3200–3100 BCE to the end of the third millennium, by two figures of the same general shape, the narrow notch and the thick notch (see Fig. 8.9 above); then, from around the twenty-seventh century BCE, by cuneiform marks of the same type, distinguished only by their respective sizes; and, finally, from the third dynasty of Ur onwards (twenty-second to twentieth century BCE), especially in the writings of Akkadian scribes, by the same vertical wedge.

In other words, as we can see from Asarhaddon's story in *The Black Stone*, and in the Assyro-Babylonian representations of the numbers 70, 80 and 90 (see Fig. 13. 23 above), the large wedge meaning 60 had evolved in line with the general evolution of cuneiform writing so as to be indistinguishable from the small wedge meaning 1.

In everyday usage, that evolution was seen as a problem, which was got round by "spelling out" 60 as *shu-shi* in numbers such as 61, 62, 63, where the confusion was potentially greatest (see Fig. 13.14 above), and eventually by replacing the sexagesimal unit with a multiple of a decimal one (Fig. 13.18 above).

But in the usage of the learned men of Mesopotamia, the graphical equivalence of the signs for 1 and 60 gave rise (at least for numbers with two orders of magnitude) to a true rule of position. As the following notations show:

SUMERIAN SYSTEM	SUMERIAN-AKKADIAN SYNTHESIS	LEARNED BABYLONIAN SYSTEM
60 + 50 + 7	60 + 50 + 7	[1 ; (50 + 7)]
60 + 60 + 40 + 1 60 + 60	60 + 60 + 40 + 1 60 + 60	[4 ; (40 + 1)]

FIG. 13.49.

Babylonian scholars realised therefore that the rule or principle could be generalised to represent all integers, provided that the old Sumerian signs for the multiples and powers of 60 were abandoned. The first to go was the 600 (= 60 × 10), for which was substituted as many chevrons (= 10) as there were 60s in the number represented. Then the sign for 3,600 (the square of 60) was dropped, and, since this number was a unit of the third sexagesimal order, it was henceforth represented by a single vertical wedge. Subsequently the sign for 36,000 was eliminated, and replaced by the sign for 10 in the position reserved for the third sexagesimal order, and so on.

For instance, instead of representing the number 1,859 by three signs for 600 followed by the notation of the number 59 (1,859 = 3 × 600 + 59), Babylonian scholars now used [30; 59] (= 30 × 60 + 59), as shown in Fig. 13.48 above, which also gives the example of the "old" and "new" representations of 4,818.

The vertical wedge thus came to represent not only the unit, but any and all powers of 60. In other words, 1 was henceforth figured by the same wedge that signified 60, 3,600, 216,000, and so on, and all 10-multiples of the base (600, 36,000, 2,160,000, etc.) by the chevron.

The discovery was extremely fruitful in itself, but, because of the very circumstances in which it arose, it gave rise to many difficulties.

THE DIFFICULTIES OF THE BABYLONIAN SYSTEM

Despite their strictly positional nature and their sexagesimal base, learned Babylonian numerals remained decimal and additive within each order of magnitude. This naturally created many ambiguous expressions and was thus the source of many errors. For example, in a mathematical text from Susa, a number [10; 15] (that is to say, $10 \times 60 + 15$, or 615) is written thus:

[10 ; 15]

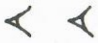
FIG. 13.50A.

However, this expression could also just as easily be read as

[25] or [10 ; 10 ; 5] $(= 10 \times 60^2 + 10 \times 60 + 5)$

FIG. 13.50B.

It is rather as if the Romans had adopted the rule of position and base 60, and had then represented expressions such as "10° 3′ 1‴" (= 36,181″) by the Roman numerals X III I, which they could easily have confused with XI II I (11° 2′ 1″), X I III (10° 1′ 3″) , and so on. Scribes in Babylon and Susa were well aware of the problem and tried to avoid it by leaving a clear space between one sexagesimal order and the next. So in the same text as the one from which Fig. 13.50 is transcribed, we find the number [10; 10] $(= 10 \times 60 + 10)$, represented as:



[10 ; 10]

FIG. 13.51.

The clear separation of the two chevrons eliminates any ambiguity with the representation of the number 20.

In another tablet from Susa the number [1; 1; 12] $(= 1 \times 60^2 + 1 \times 60 + 12)$ is written

[1 ; 1 ; 12]

FIG. 13.52A.

in which the clear separation of the leftmost wedge serves to distinguish the expression from

[2 ; 12] $(= 2 \times 60 + 12)$

FIG. 13.52B.

In some instances scribes used special signs to mark the separation of the orders of magnitude. We find double oblique wedges, or twin chevrons one on top of the other, fulfilling this role of "order separator"*:

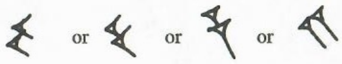


FIG. 13.53.

Here are some examples from a mathematical tablet excavated at Susa:

[1 ; 10 ; 18 ; 45] $(= 1 \times 60^3 + 10 \times 60^2 + 18 \times 60 + 45)$

Separation sign

FIG. 13.54A.

[20 ; 3 ; 13 ; 21 ; 33]

Separation sign

$(= 20 \times 60^4 + 3 \times 60^3 + 13 \times 60^2 + 21 \times 60 + 33)$

FIG. 13.54B.

The sign of separation makes the first number above quite distinct from the representation of [1; 10 + 18; 45] $(= 1 \times 60^2 + 28 \times 60 + 45)$; and for the same reason the second number above cannot be mistaken for [20 + 3; 13; 21; 33] $(= 23 \times 60^3 + 13 \times 60^2 + 21 \times 60 + 33)$.

This difficulty actually masked a much more serious deficiency of the system – the *absence of a zero*. For more than fifteen centuries, Babylonian mathematicians and astronomers worked without a concept of or sign for zero, and that must have hampered them a great deal.

In any numeral system using the rule of position, there comes a point where a special sign is needed to represent units that are missing from the number to be represented. For instance, in order to write the number *ten* using (as we now do) a decimal positional notation, it is easy enough to place the sign for 1 in second position, so as to make it signify one unit of the higher (decimal) order – but how do we signify that this sign is indeed

* In commentaries on literary texts, the same sign was used to separate head-words from their explications; in multilingual texts, the sign was used to mark the switch from one language to another; and in lists of prophecies, the sign was used to separate formulae and to mark the start of an utterance.

in second position if we have nothing to write down to mean that there is nothing in the first position? *Twelve* is easy – you put "1" in second position, and "2" in first position, itself the guarantee that the "1" is indeed in second position. But if all you have for ten is a "1" and then nothing . . . The

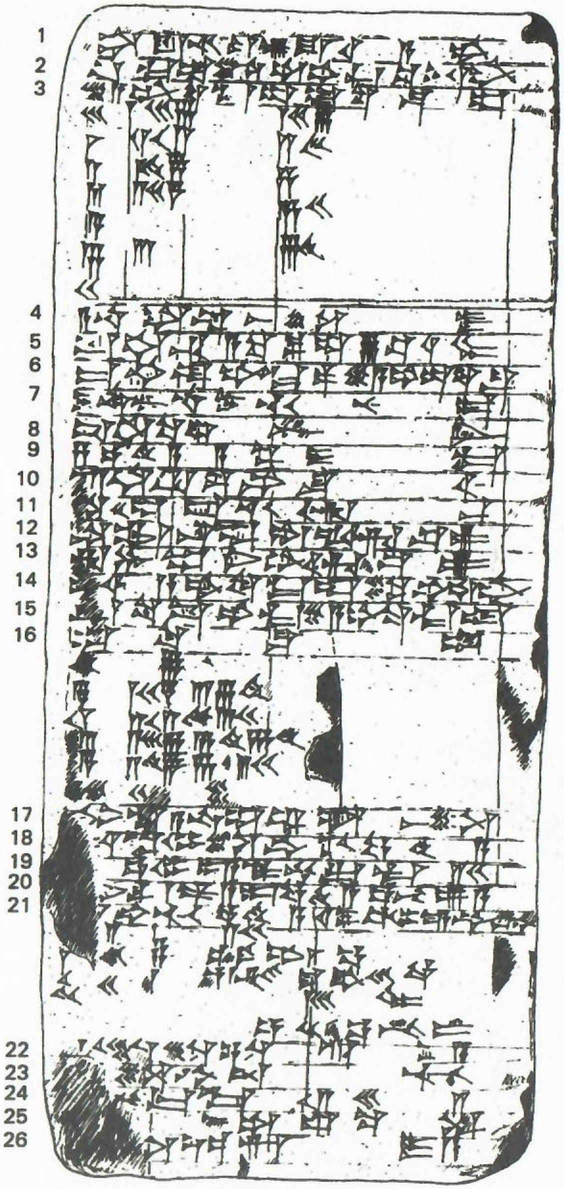


FIG. 13.55. *Important mathematical text from Larsa (Senkereh), dating from the period of the First Babylonian Dynasty (Louvre, AO 8862, side IV). See Neugebauer, tablet 38. Beneath line 16, note the representation of the number 18,144,220 as [1; 24; blank space; 3; 40].*

problem is obviously acute. Similarly, to write a number like "seven hundred and two" in a decimal positional system, you can easily put a "7" in third position and a "2" in first position, but it's not easy to tell that there's an arithmetical "nothing" between them if there is indeed *no thing* to put between them.

It became clear in the long run that such a *nothing* had to be represented by *something* if confusion in numerical calculation was to be avoided. The something that means nothing, or rather the sign that signifies the absence of units in a given order of magnitude, is, or would one day be represented by, zero.

The learned men of Babylon had no concept of zero around 1200 BCE. The proof can be seen on a tablet from Uruk (Louvre AO 17264) which gives the following solution:

"Calculate the square of  *and you get*  *"*

In decimal numbers using the rule of position, the first of these expressions (2 × 60 + 27) is equal to 147, and the square of 147 is 21,609. This latter number can be expressed in sexagesimal arithmetic as 6 × 3,600 + 0 × 60 + 9, and should therefore be written in learned Babylonian cuneiform numbers with a "9" in first position, a "6" in third position, and "nothing" in second position. If the scribe had had a concept of zero he would surely have avoided writing the square of [2; 27] as the expression [6; 9] which we see on the tablet – since the simplest resolution of [6; 9] is 6 × 60 + 9 = 369, which is not the square of 147 at all!

Another example of the same kind can be found on a Babylonian mathematical tablet from around 1700 BCE (Berlin Archaeological Museum, VAT 8528), where the numbers [2; 0; 20] (= $2 \times 60^2 + 0 \times 60 + 20$ = 7,220) and [1; 0; 10] (= $1 \times 60^2 + 0 \times 60 + 10$ = 3,610) are represented by

FIG. 13.56.

These notations are manifestly ambiguous, since they could represent, respectively, [2; 20] (= 2 × 60 + 20 = 140) and [1; 10] (= 1 × 60 + 10 = 70).

To overcome this difficulty, Babylonian scribes sometimes left a blank space in the position where there was no unit of a given order of magnitude. Here are some examples from tablets excavated at Susa (examples A, B, C) and from Fig. 13.58 below (example D, line 15). Our interpretations are not speculative, since the values given correspond to mathematical relations that are unambiguous in context:

[1 ; ↓ ; 25] (= 1 × 60² + 0 × 60 + 25)

no units of the second order

FIG. 13.57A.

[1 ; 0 ; 35] (= 1 × 60² + 0 × 60 + 35)

FIG. 13.57B.

[1 ; 0 ; 40] (= 1 × 60² + 0 × 60 + 40)

FIG. 13.57C.

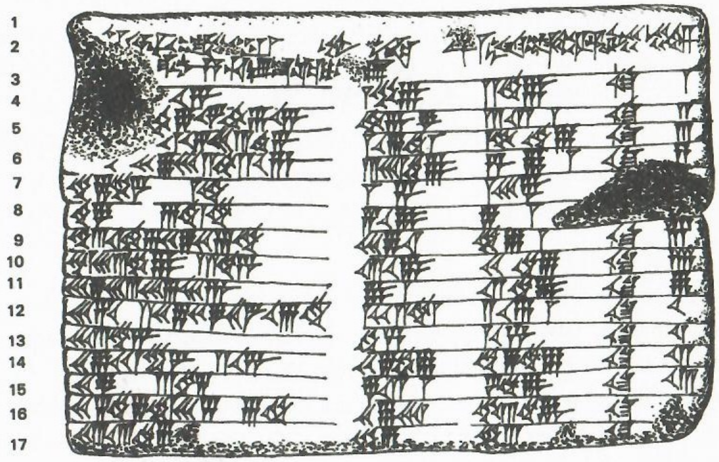
[1 ; 27 ; 0 ; 3 ; 45] (= 1 × 60⁴ + 27 × 60³ + 0 × 60² + 3 × 60 + 45)

FIG. 13.57D.

However, this did not solve the problem entirely. For a start, scribes often made mistakes or did not bother to leave the space. Secondly, the device did not allow a distinction to be made between the absence of units in one order of magnitude, and the absence of units in two or more orders of magnitude, since two spaces look much the same as one space. And finally, since the figure for 4, for instance, could mean 4 × 60, 4 × 60², 4 × 60³, or 4 × 60⁴, how could you know which order of magnitude was meant by a single expression?

These difficulties were compounded by fractions. Whereas their predecessors had given each fraction a specific sign (see Fig. 10.32 above for an example from Elam), the Babylonians used the rule of position for fractions whose denominator was a power of 60. In other words, positional sexagesimal notation was extended to what we would now call the negative powers of 60 ($60^{-1} = 1/60$, $60^{-2} = 1/60^2 = 1/3{,}600$, $60^{-3} = 1/60^3 = 1/216{,}000$, etc.). So the vertical wedge came to signify not just 1, 60, 60^2, etc., but also 1/60, 1/3,600, and so on. Two wedges could mean 2 or 120 or 1/30 or 1/1,800; the figure signifying 15 could also signify 1/4 (= 15/60), and the number 30 might just as easily mean 1/2.

Numerals were written from right to left in ascending order of the powers of 60, and from left to right in ascending negative powers of 60, exactly as we now do with our decimal positional numbering – except that in Babylon there was nothing equivalent to the decimal point that we now use to separate the integer from the fraction.



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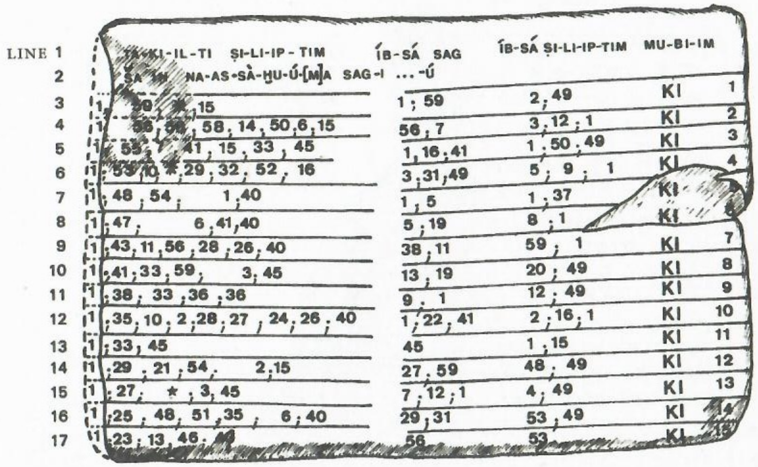

* Blank space indicating the absence of units in a given order of magnitude

FIG. 13.58. *Mathematical tablet, 1800–1700 BCE, showing that Babylonian mathematicians were already aware of the properties of right-angled triangles (Pythagoras' theorem). If we take the numbers in the leftmost column A, the second column B, and the third column C, we find that the numbers obey the relationship*

$$A = \frac{a^2}{c^2};\ B = b;\ C = c, \text{ and } a^2 = b^2 + c^2$$

This expresses the relationship by which in a right-angled triangle (with sides b and c and hypotenuse a) the square of the hypotenuse is equal to the sum of the squares on the other two sides. Columbia University, Plimpton 322. Author's own transcription