2.1.4 Summary

- Given a differentiable function $y = f(x)$, we can express the derivative of f in several different notations: $f'(x)$, $\frac{df}{dx}$, $\frac{dy}{dx}$, and $\frac{d}{dx}[f(x)]$.
- The limit definition of the derivative leads to patterns among certain families of functions that enable us to compute derivative formulas without resorting directly to the limit definition. For example, if f is a power function of the form $f(x) = x^n$, then $f'(x) = nx^{n-1}$ for any real number n other than 0. This is called the Rule for Power Functions.
- We have stated a rule for derivatives of exponential functions in the same spirit as the rule for power functions: for any positive real number a, if $f(x) = a^x$, then $f'(x) = a^x \ln(a)$.
- If we are given a constant multiple of a function whose derivative we know, or a sum of functions whose derivatives we know, the Constant Multiple and Sum Rules make it straightforward to compute the derivative of the overall function. More formally, if $f(x)$ and $g(x)$ are differentiable with derivatives $f'(x)$ and $g'(x)$ and a and b are constants, then

$$\frac{d}{dx}[af(x) + bg(x)] = af'(x) + bg'(x).$$

2.1.5 Exercises

1. Derivative of a power function.

Activate

Find the derivative of $y = x^{15/16}$.

$\frac{dy}{dx} =$ 15/(16·x^{1/16})

$y' = (x^{15/16})' = \frac{15}{16} \cdot x^{15/16 - 1}$

$= \frac{15}{16} \cdot x^{-1/16} = \frac{15}{16} \cdot \frac{1}{x^{1/16}}$

2. Derivative of a rational function. ← treat it as a power!

Activate

Find the derivative of $f(x) = \frac{1}{x^{19}}$. $= x^{-19}$

$f'(x) =$ $-19x^{-20} = \frac{-19}{x^{20}}$

$f'(x) = (x^{-19})'$

$= -19\,x^{-19-1}$

3. Derivative of a root function.

Activate

Find the derivative of

$y = \sqrt{x}$.

$\frac{dy}{dx} =$ $\frac{1}{2\sqrt{x}}$

$y = \sqrt{x} = x^{1/2}$ $y' = \frac{1}{2} x^{1/2 - 1}$

$y' = \frac{1}{2} x^{-1/2} = \frac{1}{2x^{1/2}} = \frac{1}{2\sqrt{x}}$

4. Derivative of a quadratic.

Activate

Find the derivative of $f(t) = 3t^2 - 7t + 2$.

$f'(t) =$ $6t - 7$

$f'(t) = (3t^2 - 7t + 2)'$
$= (3t^2)' - (7t)' + (2)'$
$= 3 \cdot 2t^{2-1} - 7t^{1-1} + 0$
$= 6t - 7$

5. Derivative of a sum of power functions.

Activate

Find the derivative of $y = 6t^6 - 9\sqrt{t} + \frac{7}{t}$.

$\frac{dy}{dt} =$ $36t^5 - \frac{9}{2}t^{-1/2} - 7t^{-2}$

$y' = (6t^6 - 9t^{1/2} + 7t^{-1})'$
$= (6t^6)' - (9t^{1/2})' + (7t^{-1})'$
$= 6 \cdot 6t^{6-1} - 9 \cdot \frac{1}{2}t^{\frac{1}{2}-1} + 7(-1)t^{-1-1}$

6. Simplifying a product before differentiating.

Activate

Find the derivative of $y = \sqrt{x}(x^3 + 9)$. $= x^{1/2}(x^3+9) = x^{1/2}x^3 + 9 \cdot x^{1/2}$

$= x^{7/2} + 9x^{1/2}$ $\quad y' = \frac{7}{2}x^{5/2} + 9 \cdot \frac{1}{2}x^{-1/2}$

$\frac{dy}{dx} =$ $\frac{7}{2}x^{5/2} + \frac{9}{2}x^{-1/2}$

7. Simplifying a quotient before differentiating.

Activate

Find the derivative of $y = \dfrac{x^6 + 9}{x}$. $= \frac{x^6}{x} + \frac{9}{x} = x^5 + 9x^{-1}$

$\frac{dy}{dx} =$ $5x^4 - 9x^{-2}$

$\frac{d}{dx}(y) = (x^5)' + (9x^{-1})'$
$= 5x^4 - 9x^{-2}$

8. Finding a tangent line equation.

Activate

Find an equation for the line tangent to the graph of f at $(3, 76)$, where f is given by $f(x) = 4x^3 - 4x^2 + 4$.

Have point, need slope.

$f'(x) = 12x^2 - 8x$

$f'(3) = 108 - 72 = 36$ Ready?

$y =$ $76 + 36(x-3)$

9. Determining where $f'(x) = 0$.

Activate

If $f(x) = x^3 + 6x^2 - 288x + 5$, find analytically all values of x for which $f'(x) = 0$. (Enter your answer as a comma separated list of numbers, e.g., -1,0,2)

$f'(x) = 3x^2 + 12x - 288$ Quadratic - two roots.

$\frac{-12 \pm \sqrt{144 - 4 \cdot 3 \cdot (-288)}}{6} = \frac{-12 \pm 12\sqrt{1+24}}{6} = -2 \pm 10$

$x =$ $-12, 8$

10. Let f and g be differentiable functions for which the following information is known: $f(2) = 5$, $g(2) = -3$, $f'(2) = -1/2$, $g'(2) = 2$.

a. Let h be the new function defined by the rule $h(x) = 3f(x) - 4g(x)$. Determine $h(2)$ and $h'(2)$.

$h'(x) = 3f'(x) - 4g'(x)$

$h(2) = 3f(2) - 4g(2) = 3\cdot 5 - 4\cdot(-3) = 27$

b. Find an equation for the tangent line to $y = h(x)$ at the point $(2, h(2))$.

$h'(2) = 3\cdot(-\frac{1}{2}) - 4\cdot(2) = -\frac{19}{2}$

$y = 27 - \frac{19}{2}(x - 2)$

c. Let p be the function defined by the rule $p(x) = -2f(x) + \frac{1}{2}g(x)$. Is p increasing, decreasing, or neither at $a = 2$? Why?

$p'(x) = -2f'(x) + \frac{1}{2}g'(x)$

$p(2) = -2\cdot 5 + \frac{1}{2}(-3) = -\frac{23}{2}$

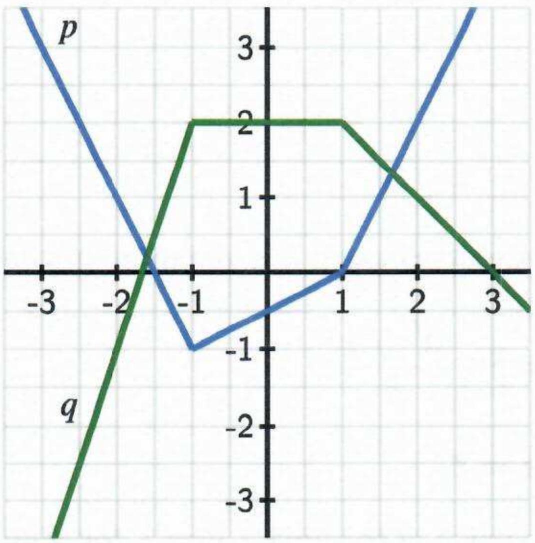
d. Estimate the value of $p(2.03)$ by using the local linearization of p at the point $(2, p(2))$.

$p'(2) = -2(-\frac{1}{2}) + \frac{1}{2}\cdot 2 = 2$

$L(x) = -\frac{23}{2} + 2(x - 2)$

$L(2.03) = -\frac{23}{2} + 2(2.03 - 2) = -\frac{23}{2} + 0.06 = -11.44$

11. Let functions p and q be the piecewise linear functions given by their respective graphs in Figure 2.1.6. Use the graphs to answer the following questions.



$p'(x) = \begin{cases} -2 & x < -1 \\ \frac{1}{2} & x \in (-1, 1) \\ 2 & x > 1 \end{cases}$

$q'(x) = \begin{cases} 3 & x < -1 \\ 0 & x \in (-1, 1) \\ -1 & x > 1 \end{cases}$

$r(x) = p(x) + 2q(x)$

$r'(x) = p'(x) + 2q'(x)$

Figure 2.1.6. The graphs of p (in blue) and q (in green).

a. At what values of x is p not differentiable? At what values of x is q not differentiable? Why? Corners! no tangent lines!

$p: x \in \{-1, 1\}$

$q: x \in \{-1, 1\}$

b. Let $r(x) = p(x) + 2q(x)$. At what values of x is r not differentiable? Why?

Still has corners at $x \in \{-1, 1\}$

c. Determine $r'(-2)$ and $r'(0)$.

$r'(-2) = p'(-2) + 2q'(-2) = -2 + 2\cdot 3 = 4$

$r'(0) = \frac{1}{2} + 2\cdot 0 = \frac{1}{2}$

d. Find an equation for the tangent line to $y = r(x)$ at the point $(2, r(2))$.

$y = 4 + \frac{1}{2}(x - 2)$

$r(2) = p(2) + 2q(2) = 2 + 2\cdot 1 = 4$

12. Consider the functions $r(t) = t^t$ and $s(t) = \arccos(t)$, for which you are given the facts that $r'(t) = t^t(\ln(t) + 1)$ and $s'(t) = -\frac{1}{\sqrt{1-t^2}}$. Do not be concerned with where these derivative formulas come from. We restrict our interest in both functions to the domain $0 < t < 1$.

a. Let $w(t) = 3t^t - 2\arccos(t)$. Determine $w'(t)$.