Homework problems, Section 2.2

1. Suppose that $V(t) = 24 \cdot 1.07^t + 6\sin(t)$ represents the value of a person's investment portfolio in thousands of dollars in year t, where $t = 0$ corresponds to January 1, 2010.

a. At what instantaneous rate is the portfolio's value changing on January 1, 2012? Include units on your answer.

b. Determine the value of $V''(2)$. What are the units on this quantity and what does it tell you about how the portfolio's value is changing?

c. On the interval $0 \le t \le 20$, graph the function $V(t) = 24 \cdot 1.07^t + 6\sin(t)$ and describe its behavior in the context of the problem. Then, compare the graphs of the functions $A(t) = 24 \cdot 1.07^t$ and $V(t) = 24 \cdot 1.07^t + 6\sin(t)$, as well as the graphs of their derivatives $A'(t)$ and $V'(t)$. What is the impact of the term $6\sin(t)$ on the behavior of the function $V(t)$?

```
In[172]:= V[t_] := 24 * 1.07^t + 6 Sin[t] (* where t=0 corresponds to January 1,2010. *)

V'[t]
Out[173]= 1.62380756337156 × 1.07^t + 6 Cos[t]
```

a. At what instantaneous rate is the portfolio's value changing on January 1, 2012? Include units on your answer.

```
V'[2] (* thousands of dollars/year *)
Out[174]= -0.637783739978759
```

b. Determine the value of V''(2). What are the units on this quantity and what does it tell you about how the portfolio's value is changing?

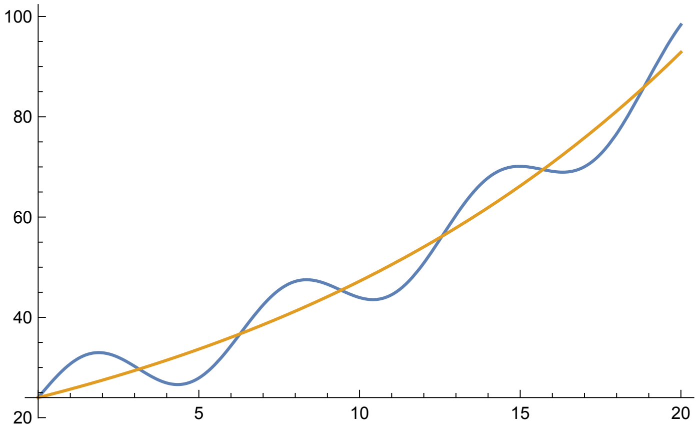
```
In[175]:= V''[2] (* thousands of dollars/year/year *)
Out[175]= -5.33000055165503
```

Since it's negative, the slope (already negative) is headed even more negative....

c. graph the function

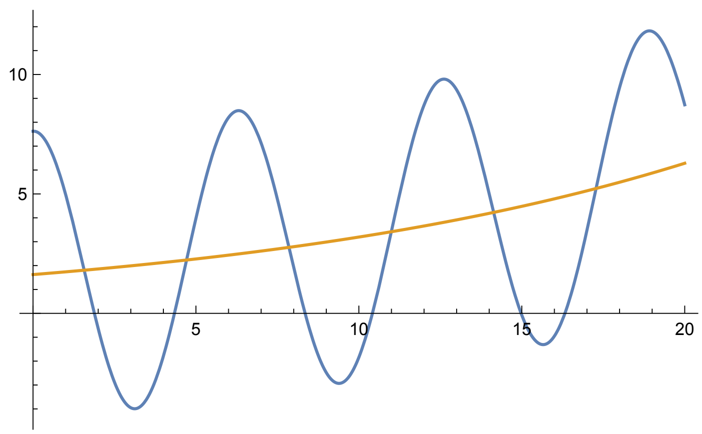
In[176]:=
```
A[t_] := 24 * 1.07^t
Plot[{V[t], A[t]}, {t, 0, 20}]
```

Out[177]=



In[178]:=
```
Plot[{V'[t], A'[t]}, {t, 0, 20}]
```

Out[178]=



The general trend is that of an increasing exponential, but there appears to be a seasonal trend to fluctuate sinusoidally (the 6sin(t) part).

2. Let $f(x) = 3\cos(x) - 2\sin(x) + 6$.

a. Determine the exact slope of the tangent line to $y = f(x)$ at the point where $a = \frac{\pi}{4}$.

b. Determine the tangent line approximation to $y = f(x)$ at the point where $a = \pi$.

c. At the point where $a = \frac{\pi}{2}$, is f increasing, decreasing, or neither?

d. At the point where $a = \frac{3\pi}{2}$, does the tangent line to $y = f(x)$ lie above the curve, below the curve, or neither? How can you answer this question without even graphing the function or the tangent line?

In[181]:=
```
f[x_] := 3 Cos[x] - 2 Sin[x] + 6
```

a. Determine the exact slope of the tangent line to y=f(x) at the point where a=Pi/4.

In[184]:=
```
f'[x]
f'[Pi / 4]
N[%]
```

Out[184]= $-2\,\text{Cos}[x] - 3\,\text{Sin}[x]$

Out[185]= $-\frac{3}{\sqrt{2}} - \sqrt{2}$

Out[186]= -3.53553390593274

b. Determine the tangent line approximation to y=f[x] at the point where a=Pi.

In[189]:=
```
a = Pi;
l[x_] := f[a] + f'[a] (x - a)
l[x]
```

Out[191]= $3 + 2\,(-\pi + x)$

c. At the point where a = Pi/2, is f increasing, decreasing, or neither?

In[192]:=
```
f'[Pi / 2]
```

Out[192]= -3

decreasing.

d. At the point where a = 3Pi/2, does the tangent line to y=f(x) lie above the curve, below the curve, or neither? How can you answer this question without even graphing the function or the tangent line?

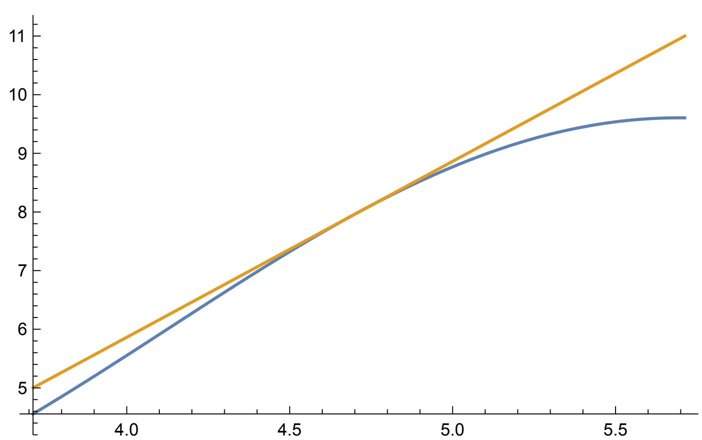
In[193]:=
```
f''[3 Pi / 2]
```

Out[193]= -2

Concave down, so the line is above the function, as we can see in the plot:

In[195]:=
```
a = 3 Pi / 2;
l[x_] := f[a] + f'[a] (x - a)
Plot[{f[x], l[x]}, {x, a - 1, a + 1}]
```

Out[197]=



3. In this exercise, we explore how the limit definition of the derivative more formally shows that $\frac{d}{dx}[\sin(x)] = \cos(x)$. Letting $f(x) = \sin(x)$, note that the limit definition of the derivative tells us that

$$f'(x) = \lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h}.$$

a. Recall the trigonometric identity for the sine of a sum of angles α and β: $\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$. Use this identity and some algebra to show that

$$f'(x) = \lim_{h \to 0} \frac{\sin(x)(\cos(h) - 1) + \cos(x)\sin(h)}{h}.$$

b. Next, note that as h changes, x remains constant. Explain why it therefore makes sense to say that

$$f'(x) = \sin(x) \cdot \lim_{h \to 0} \frac{\cos(h) - 1}{h} + \cos(x) \cdot \lim_{h \to 0} \frac{\sin(h)}{h}.$$

c. Finally, use small values of h to estimate the values of the two limits in (c):

$$\lim_{h \to 0} \frac{\cos(h) - 1}{h} \quad \text{and} \quad \lim_{h \to 0} \frac{\sin(h)}{h}.$$

d. What do your results in (b) and (c) thus tell you about $f'(x)$?

e. By emulating the steps taken above, use the limit definition of the derivative to argue convincingly that $\frac{d}{dx}[\cos(x)] = -\sin(x)$.

Since we did this in class, and included a little more detail (e.g. by replacing the limits by forms that looked even more like derivatives), I'm going to leave this to you! But feel free to ask me for more details if you wish.