particular, if $f(a) = g(a) = 0$ and f and g are differentiable at a, L'Hôpital's Rule tells us that

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

- When we write $x \to \infty$, this means that x is increasing without bound. Thus, $\lim_{x\to\infty} f(x) = L$ means that we can make $f(x)$ as close to L as we like by choosing x to be sufficiently large. Similarly, $\lim_{x\to a} f(x) = \infty$, means that we can make $f(x)$ as large as we like by choosing x sufficiently close to a.
- A version of L'Hôpital's Rule also helps us evaluate indeterminate limits of the form $\frac{\infty}{\infty}$. If f and g are differentiable and both approach zero or both approach $\pm\infty$ as $x \to a$ (where a is allowed to be ∞), then

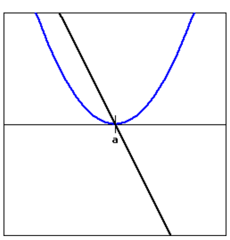
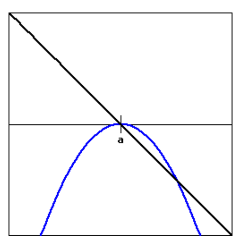
$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

2.8.4 Exercises

1. L'Hôpital's Rule with graphs.

Activate

For the figures below, determine the nature of $\lim_{x\to a} \frac{f(x)}{g(x)}$, if $f(x)$ is shown as the blue curve and $g(x)$ as the black curve.

	
$\lim_{x\to a} \frac{f(x)}{g(x)} =$ $\approx \frac{(x-a)^2}{m(x-a)} \to 0$ ▪ positive ▪ negative ▪ zero ▪ undefined	$\lim_{x\to a} \frac{f(x)}{g(x)} =$ $\approx \frac{-(x-a)^2}{m(x-a)} \to 0$ ▪ positive ▪ negative ▪ zero ▪ undefined

$f'(x) = \frac{1}{4} \cdot \frac{1}{(x/4)} = \frac{1}{x}$

2. L'Hôpital's Rule to evaluate a limit.

Activate

$g'(x) = 2x$

Find the limit: $\lim_{x\to 4} \frac{\ln(x/4)}{x^2 - 16} =$ $\frac{f'(4)}{g'(4)} = \frac{1/4}{8} = \boxed{\frac{1}{32}}$

*(Enter **undefined** if the limit does not exist.)*

3. Determining if L'Hôpital's Rule applies.

Activate

Compute the following limits using l'H\^opital's rule if appropriate. Use INF to denote ∞ and MINF to denote $-\infty$.

$$\lim_{x\to 0}\frac{1-\cos(7x)}{1-\cos(3x)} = \underline{\quad\quad\quad} \quad \lim_{x\to 0}\frac{7\sin(7x)}{3\sin 3x} = \lim_{x\to 0}\frac{-49\cos(7x)}{-9\cos 3x} = \frac{-49}{-9} = \boxed{\frac{49}{9}}$$

$$\lim_{x\to 1}\frac{4^x-3^x-1}{x^2-1} = \underline{\quad\quad\quad} \quad \lim_{x\to 1}\frac{\ln 4\cdot 4^x - \ln 3\cdot 3^x}{2x} = \frac{4\ln 4 - 3\ln 3}{2}$$

4. Using L'Hôpital's Rule multiple times.

Activate

Evaluate the limit using L'Hopital's rule.

$$\lim_{x\to\infty}\frac{15x^3}{e^{2x}} = \underline{\ 0\ }$$ help (limits) [1]

$$\lim_{x\to\infty}\frac{45x^2}{2e^{2x}} = \lim_{x\to\infty}\frac{90x}{4e^{2x}} = \lim_{x\to\infty}\frac{90}{8e^{2x}}$$

5. Let f and g be differentiable functions about which the following information is known: $f(3)=g(3)=0$, $f'(3)=g'(3)=0$, $f''(3)=-2$, and $g''(3)=1$. Let a new function h be given by the rule $h(x)=\frac{f(x)}{g(x)}$. On the same set of axes, sketch possible graphs of f and g near $x=3$, and use the provided information to determine the value of

$$\lim_{x\to 3} h(x). = \frac{f''(3)}{g''(3)} = \frac{-2}{1} = \boxed{-2}$$

See below, at bottom

Provide explanation to support your conclusion.

6. Find all vertical and horizontal asymptotes of the function

$y = 3/5$

$x = c$

$$R(x)=\frac{3(x-a)(x-b)}{5(x-a)(x-c)}, = \frac{3(x-b)}{5(x-c)} \quad \text{w/ } x\neq a$$

There's a hole in the graph at $x=a$.

where a, b, and c are distinct, arbitrary constants. In addition, state all values of x for which R is not continuous. Sketch a possible graph of R, clearly labeling the values of a, b, and c.

See below, bottom

7. Consider the function $g(x)=x^{2x}$, which is defined for all $x>0$. Observe that $\lim_{x\to 0^+} g(x)$ is indeterminate due to its form of 0^0. (Think about how we know that $0^k=0$ for all $k>0$, while $b^0=1$ for all $b\neq 0$, but that neither rule can apply to 0^0.)

a. Let $h(x)=\ln(g(x))$. Explain why $h(x)=2x\ln(x)$.

b. Next, explain why it is equivalent to write $h(x)=\frac{2\ln(x)}{\frac{1}{x}}$.

c. Use L'Hôpital's Rule and your work in (b) to compute $\lim_{x\to 0^+} h(x)$.

d. Based on the value of $\lim_{x\to 0^+} h(x)$, determine $\lim_{x\to 0^+} g(x)$.

8. Recall we say that function g ***dominates*** function f provided that $\lim_{x\to\infty} f(x) = \infty$, $\lim_{x\to\infty} g(x) = \infty$, and $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$.

a. Which function dominates the other: $\ln(x)$ or $\sqrt{x}$?

$\lim_{x\to\infty} \frac{\ln(x)}{\sqrt{x}} = \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{2}x^{-1/2}} = \frac{2}{x^{1/2}} \to 0$

b. Which function dominates the other: $\ln(x)$ or $\sqrt[n]{x}$? (n can be any positive integer)

$\frac{\ln(x)}{x^{1/n}} \sim \frac{1/x}{x^{1/n}} \to 0$

c. Explain why e^x will dominate any polynomial function.

$\lim_{x\to\infty} \frac{p(x)}{e^x} = 0$ deriv → 0. deriv = e^x

d. Explain why x^n will dominate $\ln(x)$ for any positive integer n.

Just take derivatives; $\frac{1}{x}$ grows, but more slowly

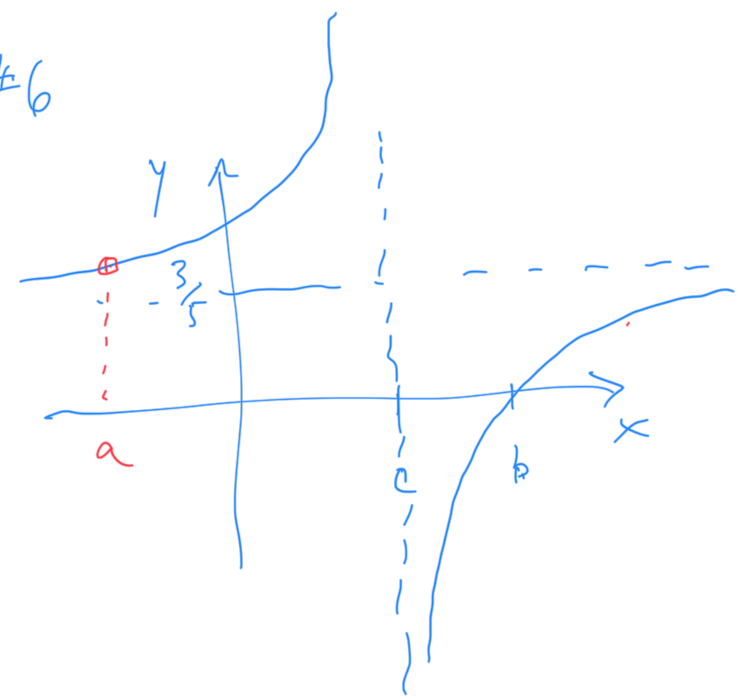
$n \frac{1}{x^n} \to 0$

e. Give any example of two nonlinear functions such that neither dominates the other.

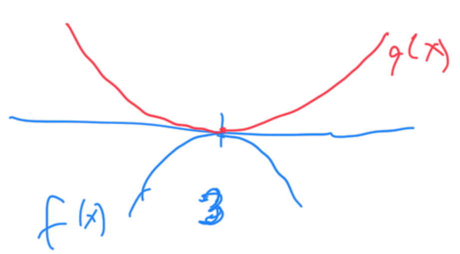
$3x^2, 4x^2$

R is not continuous at $x = a$ & $x = c$.

#6



#5 e.g. $g(x) = \frac{1}{2}(x-3)^2$



e.g. $f(x) = -(x-3)^2$

$f'(3) = -2(3-3) = 0$

$f''(3) = -2$