Section 3.4 Homework

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Problems 1-9

1. Maximizing the volume of a box.

Activate

An open box is to be made out of a 10-inch by 18-inch piece of cardboard by cutting out squares of equal size from the four corners and bending up the sides. Find the dimensions of the resulting box that has the largest volume.

Dimensions of the bottom of the box: _________ x _________

Height of the box: _________

Observations: the height must be less than five inches, since that keeps the width positive; and the height must be positive, so the domain is [0,5].

So in the end the solution is that the height is a little over 2 inches (approx 2.1 inches), and the

In[1236]:=
```
Clear[l, w, h, s];
volume[l_, w_, h_] := l*w*h
w[h_] = 10 - 2 h
l[h_] := 18 - 2 h
solns = Solve[D[volume[l[h], w[h], h], h] == 0, h]
N[solns]
height = 1/3 (14 - Sqrt[61])
N[{w[height], l[height]}]
```

Out[1238]= $10 - 2h$

Out[1240]= $\left\{\left\{h \to \frac{1}{3}\left(14 - \sqrt{61}\right)\right\}, \left\{h \to \frac{1}{3}\left(14 + \sqrt{61}\right)\right\}\right\}$

Out[1241]= {{h → 2.06325010803112}, {h → 7.27008322530222}}

Out[1242]= $\frac{1}{3}\left(14 - \sqrt{61}\right)$

Out[1243]= {5.87349978393777, 13.8734997839378}

2. Minimizing the cost of a container.

Activate

A rectangular storage container with an open top is to have a volume of 26 cubic meters. The length of its base is twice the width. Material for the base costs 11 dollars per square meter. Material for the sides costs 9 dollars per square meter. Find the cost of materials for the cheapest such container.

Total cost = ____________________________ (Round to the nearest penny and include monetary units. For example, if your answer is 1.095, enter $1.10 including the dollar sign and second decimal place.)

Observations:

total volume is 26 cubic meters (damned big box!)

length of base is twice the width: l=2w

volume=2w^2h = 26, so h = 13/(w^2)

Thus

total cost = base cost + sides cost

base costs 11 dollars per square meter; sides cost only 9 dollars per square meter.

cost = 11*l*w + 9[2(long side cost)+2(short side cost)]= 11*l*w + 9[2(l*h)+2(w*h)]

Only one real solution, w=3 $\left(\frac{13}{22}\right)^{1/3}$

It has to be a minimum, because we can this cost infinitely big by taking w too small or too large.

Total cost: $418.28

In[1244]:=
```
Clear[l, w, h, s];
l = 2 w;
h = 13 / w^2;
cost[w_] = 11 * l * w + 9 (2 (l * h) + 2 (w * h))
solns = Solve[cost'[w] == 0, w]
N[solns]
N[cost[3 (13/22)^(1/3)]]
```

Out[1247]= $\frac{702}{w} + 22 w^2$

Out[1248]= $\left\{\left\{w \to -3 \left(-\frac{13}{22}\right)^{1/3}\right\}, \left\{w \to 3 \left(\frac{13}{22}\right)^{1/3}\right\}, \left\{w \to 3 (-1)^{2/3} \left(\frac{13}{22}\right)^{1/3}\right\}\right\}$

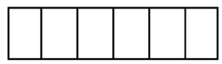
Out[1249]= {{w → -1.25872681121724 - 2.18017878987741 i},
{w → 2.51745362243447}, {w → -1.25872681121724 + 2.18017878987741 i}}

Out[1250]= 418.279800913158

3. Maximizing area contained by a fence.

Activate

An ostrich farmer wants to enclose a rectangular area and then divide it into six pens with fencing parallel to one side of the rectangle (see the figure below). There are 620 feet of fencing available to complete the job. What is the largest possible total area of the six pens?



Largest area = ___________________ (include help (units) [4])

Observations:

Let's say that the bounding rectangle has dimensions length and width.

The perimeter P = 620 feet.

It can be represented as 2length+7width, as well.

The total area is width*length.

In[1251]:=
```
Clear[l, w, h, s, p];
p = 2 l + 7 w;
solns = Solve[p == 620, l]
l = l /. solns[[1]][[1]];
area[w_] = l * w
solns = Solve[area'[w] == 0, w]
w = w /. solns[[1]][[1]];
N[{w, l}]
N[area[w]]
Plot[area[w], {w, 0, 60}]
```

Out[1253]= $\left\{\left\{l \to \frac{1}{2}(620 - 7w)\right\}\right\}$

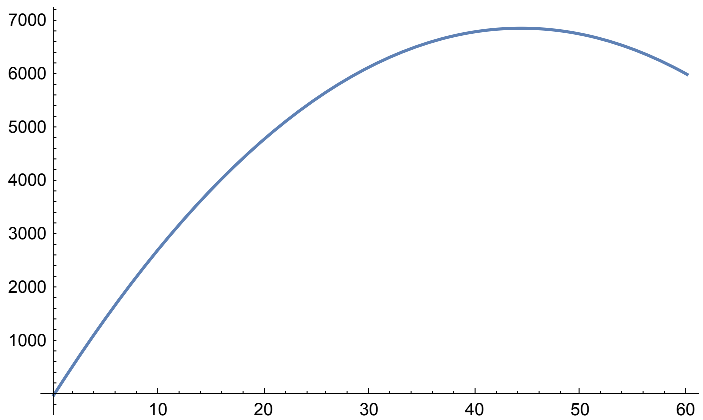
Out[1255]= $\frac{1}{2}(620 - 7w)\,w$

Out[1256]= $\left\{\left\{w \to \frac{310}{7}\right\}\right\}$

Out[1258]= {44.2857142857143, 155.}

Out[1259]= 6864.28571428572

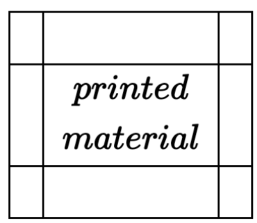
Out[1260]=



4. Minimizing the area of a poster.

Activate

The top and bottom margins of a poster are 8 cm and the side margins are each 6 cm. If the area of printed material on the poster is fixed at 388 square centimeters, find the dimensions of the poster with the smallest area.



Width = ________________________ (include help (units) 5)

Height = ________________________ (include help (units) 6)

Observations:
The whole paper has dimensions length times width (l and w).
The printed area has length and width reduced by twice the respective margins, and its area is 388 cm^2.
So we solve for l as a function of w, and then we have the total area in terms of w.

We differentiate and find two solutions, one of them negative. So the other is our animal, with dimensions of cm.

In[1261]:=
```
Clear[l, w, h, s, p];
a = (l - 2 * 8) (w - 2 * 6);
solns = Solve[a == 388, l]
l = l /. solns[[1]][[1]];
area[w_] = l * w
solns = Solve[area'[w] == 0, w]
w = w /. solns[[2]][[1]]
N[{w, l}]
N[area[w]]
Plot[area[w], {w, 12, 60}]
```

Out[1263]= $\left\{\left\{l \to \frac{4\,(49 + 4\,w)}{-12 + w}\right\}\right\}$

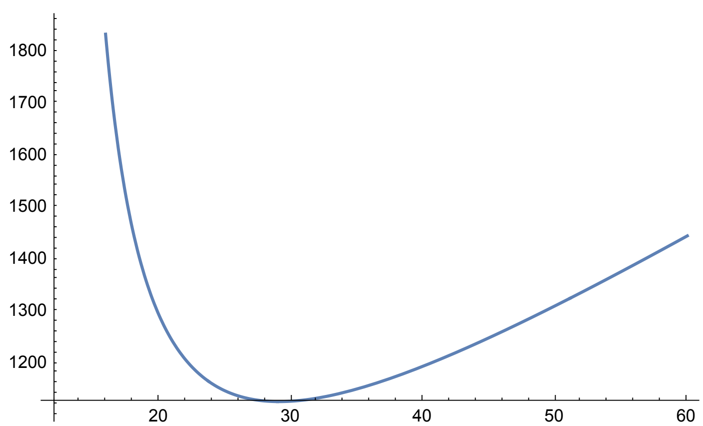
Out[1265]= $\frac{4\,w\,(49 + 4\,w)}{-12 + w}$

Out[1266]= $\left\{\left\{w \to 12 - \sqrt{291}\right\}, \left\{w \to 12 + \sqrt{291}\right\}\right\}$

Out[1267]= $12 + \sqrt{291}$

Out[1268]= {29.058722109232, 38.7449628123093}

Out[1269]= 1125.87910749542

Out[1270]=


5. Maximizing the area of a rectangle.

Activate

A rectangle is inscribed with its base on the x-axis and its upper corners on the parabola $y = 1 - x^2$. What are the dimensions of such a rectangle with the greatest possible area?

Width = ____________

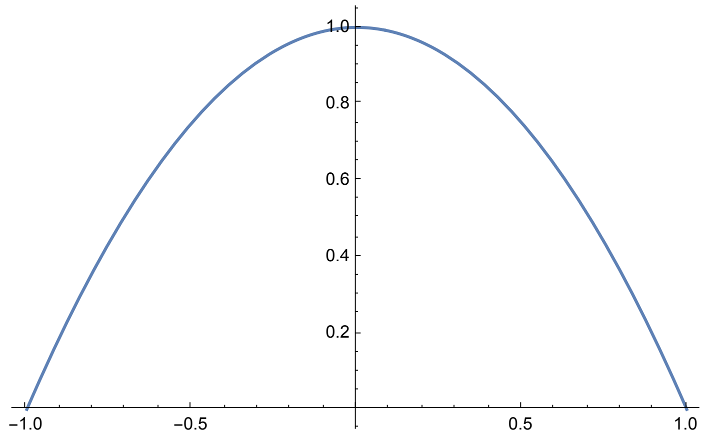
Height = ____________

Observations:

This parabola has roots at + and - 1, so the base will live there. The rectangle will be symmetrically located about the y-axis. So the base is of length 2x, and the height (1-x^2).

In[1271]:=
```
Plot[1 - x^2, {x, -1, 1}]
```

Out[1271]=



In[1272]:=
```
Clear[l, w, h, s, p];
area[x_] = x (1 - x^2)
solns = Solve[area'[x] == 0, x]
ex = 1 / Sqrt[3]
l = 2 ex
h = 1 - (ex) ^2
N[{l, h}]
N[area[ex]]
```

Out[1273]= $x\left(1-x^2\right)$

Out[1274]= $\left\{\left\{x\to-\frac{1}{\sqrt{3}}\right\},\left\{x\to\frac{1}{\sqrt{3}}\right\}\right\}$

Out[1275]= $\frac{1}{\sqrt{3}}$

Out[1276]= $\frac{2}{\sqrt{3}}$

Out[1277]= $\frac{2}{3}$

Out[1278]= {1.15470053837925, 0.666666666666667}

Out[1279]= 0.384900179459751

6. A rectangular box with a square bottom and closed top is to be made from two materials. The material for the sides costs $1.50 per square foot and the material for the top and bottom costs $3.00 per square foot. If you are willing to spend $15 on the box, what is the largest volume it can contain? Justify your answer completely using calculus.

Observations:
length = width: l=w
Largest volume: 1.52145154862546 cubic feet

In[1280]:=
```
Clear[l, w, h, s];
cost[w_, h_] = 3 * (2 * (w * w)) + 1.5 (2 (w * h) + 2 (w * h))
solns = Solve[cost[w, h] == 15, h]
h[w_] = h /. solns[[1]][[1]]
volume[w_, h_] = w * w * h
solns = Solve[D[volume[w, h[w]], w] == 0, w]
width = w /. solns[[2]][[1]]
height = h[width]
volume[width, height]
```

Out[1281]= $6.\, h\, w + 6\, w^2$

Out[1282]= $\left\{\left\{h \to \frac{0.166666666666667\,(15. - 6.\, w^2)}{w}\right\}\right\}$

Out[1283]= $\frac{0.166666666666667\,(15. - 6.\, w^2)}{w}$

Out[1284]= $h\, w^2$

Out[1285]= $\{\{w \to -0.912870929175277\}, \{w \to 0.912870929175277\}\}$

Out[1286]= 0.912870929175277

Out[1287]= 1.82574185835055

Out[1288]= 1.52145154862546

7. A farmer wants to start raising cows, horses, goats, and sheep, and desires to have a rectangular pasture for the animals to graze in. However, no two different kinds of animals can graze together. In order to minimize the amount of fencing she will need, she has decided to enclose a large rectangular area and then divide it into four equally sized pens by adding three segments of fence inside the large rectangle that are parallel to two existing sides. She has decided to purchase 7500 ft of fencing. What is the maximum possible area that each of the four pens will enclose?

Observations: this one’s exactly like #3, with different parameters. Asks a slightly different question, however: What is the maximum possible area for each pen?

Since there are four pens, 351562.5 square feet.

In[1289]:=
```
Clear[l, w, h, s, p];
p = 2 l + 5 w;
solns = Solve[p == 7500, l]
l = l /. solns[[1]][[1]];
area[w_] = l * w
solns = Solve[area'[w] == 0, w]
w = w /. solns[[1]][[1]];
N[{w, l}]
N[area[w] / 4]
Plot[area[w] / 4, {w, 0, 1000}]
```

Out[1291]= $\left\{\left\{l \to -\frac{5}{2}(-1500 + w)\right\}\right\}$

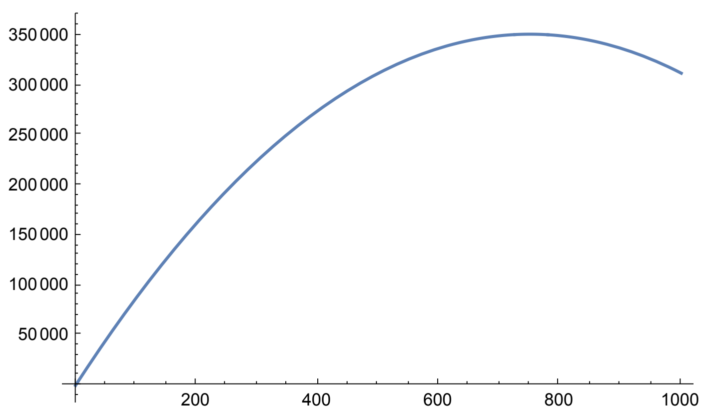
Out[1293]= $-\frac{5}{2}(-1500 + w)\, w$

Out[1294]= {{w → 750}}

Out[1296]= {750., 1875.}

Out[1297]= 351 562.5

Out[1298]=



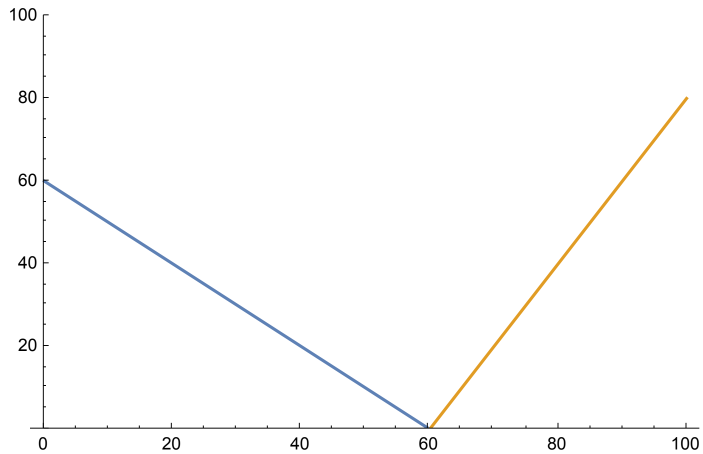
8. Two vertical poles of heights 60 ft and 80 ft stand on level ground, with their bases 100 ft apart. A cable that is stretched from the top of one pole to some point on the ground between the poles, and then to the top of the other pole. What is the minimum possible length of cable required? Justify your answer completely using calculus.

Observations:
Let's call the point on the ground to which the cables attach x.
Then the total length of the cable is the sum of two hypotenuses:
length=Sqrt[60^2+x^2]+Sqrt[(100-x)^2+80^2]

The minimum length ends up being 172.046505340853 ft.

In[1299]:=
```
Plot[{60 - x, 2 (x - 60)}, {x, 0, 100}, PlotRange → {0, 100}]
length[x_] = Sqrt[60^2 + x^2] + Sqrt[(100 - x)^2 + 80^2]
solns = Solve[length'[x] == 0, x]
location = x /. solns[[1]][[1]]
N[{location, length[location]}]
```

Out[1299]=



Out[1300]= $\sqrt{6400 + (100 - x)^2} + \sqrt{3600 + x^2}$

Out[1301]= $\left\{\left\{x \to \frac{300}{7}\right\}\right\}$

Out[1302]= $\frac{300}{7}$

Out[1303]= {42.8571428571429, 172.046505340853}