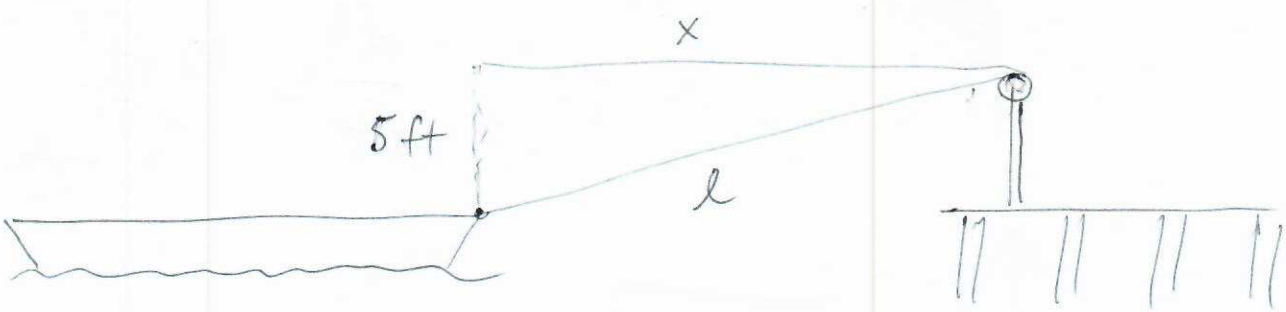
#4



$$\frac{dl}{dt} = -2\ ft/s$$

Rate at which boat approaches dock $= \frac{dx}{dt}$

Relate those rates! Easy:

$5^2 + x^2 = l^2$

Differentiate both sides:

$0 + 2x\frac{dx}{dt} = 2l\frac{dl}{dt}$

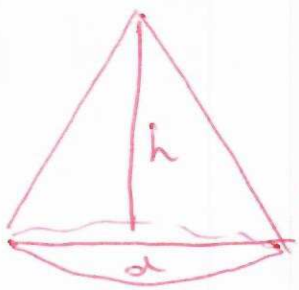
when $l = 13$ ft, $x = \sqrt{13^2 - 5^2} = \underline{\underline{12\ ft}}$

$+\ \frac{dl}{dt} = -2.$

$\therefore\ 24\frac{dx}{dt} = -4 \cdot 13$ $\boxed{\frac{dx}{dt} = -13/6\ ft/s}$

$\simeq 2.2\ ft/s$

$$\frac{dV}{dt} = 10\ \frac{ft^3}{\text{minute}}$$



$\boxed{h = d}$ (given) $\left(r = \frac{d}{2} = \frac{h}{2} \right)$

How fast is the height of the pile increasing when the pile is 23 ft high?

$\boxed{\text{Asking for } \frac{dh}{dt}}$

We want to relate $\frac{dV}{dt}$ & $\frac{dh}{dt}$;

looking for a relation between V & h, & differentiate wrt time t.

$$\boxed{V = \frac{1}{3}\pi r^2 h}$$

$$V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 \cdot h = \frac{1}{12}\pi h^3$$

$$\therefore \boxed{V(h(t)) = \frac{1}{12}\pi\left(h(t)\right)^3}$$

$$\frac{dV}{dt} = \left[\frac{1}{12}\pi\left(h(t)^3\right)\right]'$$

$$= \frac{\pi}{12}\left[h(t)^3\right]'$$

$$= \frac{\pi}{12} \cdot 3h(t)^2 \cdot \frac{dh}{dt}$$

$$\boxed{\frac{dV}{dt} = \frac{\pi}{4} h^2 \frac{dh}{dt}}$$ related rates

When $h = 23$ ft,

$$\frac{dh}{dt} = \frac{1}{\frac{\pi}{4}h^2}\frac{dV}{dt} = \frac{1}{\frac{\pi}{4}(23)^2} \cdot 10 \frac{ft^3}{minute}$$

$$= \frac{40}{\pi(23)^2} \frac{ft}{minute}$$

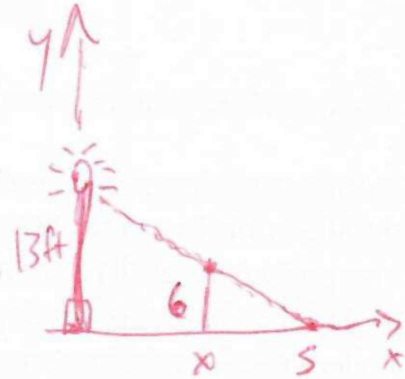
$$= 0.024 \, ft/min.$$

3.5.3 Exercises

1. Height of a conical pile of gravel.

Activate

Gravel is being dumped from a conveyor belt at a rate of 10 cubic feet per minute. It forms a pile in the shape of a right circular cone whose base diameter and height are always the same. How fast is the height of the pile increasing when the pile is 23 feet high? Recall that the volume of a right circular cone with height h and radius of the base r is given by $V = \frac{1}{3}\pi r^2 h$.



When the pile is 23 feet high, its height is increasing at ________ feet per minute.

2. Movement of a shadow. $\frac{dx}{dt} = 6 \frac{ft}{s}$

Activate

A street light is at the top of a 13 foot tall pole. A 6 foot tall woman walks away from the pole with a speed of 6 ft/sec along a straight path. How fast is the tip of her shadow moving when she is 30 feet from the base of the pole?

The tip of the shadow is moving at ________________ ft/sec.

Relate $\frac{dx}{dt}$ + $\frac{ds}{dt}$

$$\underbrace{\frac{s-x}{6} = \frac{s}{13}}_{\text{similar triangles.}}$$

$$\frac{s}{6} - \frac{s}{13} = \frac{x}{6}$$

$$7s = 13x \qquad s = \frac{13}{7}x$$

$$\frac{ds}{dt} = \frac{13}{7}\frac{dx}{dt} = \frac{13}{7}\cdot 6\frac{ft}{s} = \frac{78}{7}\frac{ft}{s}$$

$(\approx 11 \frac{ft}{s})$

3. A leaking conical tank.

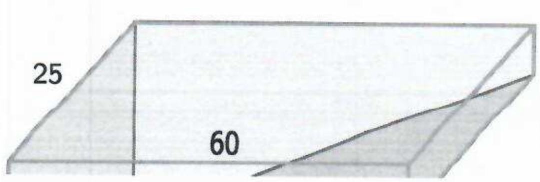
Activate

Water is leaking out of an inverted conical tank at a rate of 9600.0 cm^3/min at the same time that water is being pumped into the tank at a constant rate. The tank has height 7.0 m and the the diameter at the top is 5.0 m. If the water level is rising at a rate of 22.0 cm/min when the height of the water is 1.5 m, find the rate at which water is being pumped into the tank in cubic centimeters per minute.

Answer: ________________________ cm^3/min

4. A sailboat is sitting at rest near its dock. A rope attached to the bow of the boat is drawn in over a pulley that stands on a post on the end of the dock that is 5 feet higher than the bow. If the rope is being pulled in at a rate of 2 feet per second, how fast is the boat approaching the dock when the length of rope from bow to pulley is 13 feet?

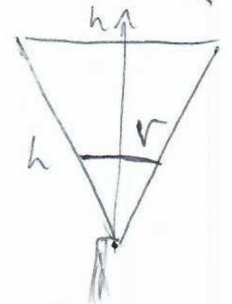
5. A swimming pool is 60 feet long and 25 feet wide. Its depth varies uniformly from 3 feet at the shallow end to 15 feet at the deep end, as shown in the Figure 3.5.5.



#3 $\quad$ 1

$D = 5\,m = 2R$

$H = 7\,m$



$$\frac{dV_{out}}{dt} = -9600 \ \text{cm}^3/\text{min}$$

$$\frac{dV_{in}}{dt} = \text{constant.}$$

$$\left.\frac{dh}{dt}\right|_{1.5m} = 22 \ \text{cm}/\text{min}$$

Relate $\frac{dh}{dt}$ & $\frac{dV_{in}}{dt}$

$$\frac{dV}{dt} = \frac{dV_{out}}{dt} + \frac{dV_{in}}{dt}$$

$$= -9600 + \frac{dV_{in}}{dt}$$

$$V = \frac{1}{3}\pi r^2 h$$

Similar triangles : $\frac{r}{h} = \frac{R}{H} \rightarrow r = \frac{R}{H} h = \frac{5/2}{7} h$

$$= \frac{5}{14} h$$

$\therefore$ $V = \frac{1}{3}\pi\left(\frac{5}{14}h\right)^2 h$

$$= \underbrace{\frac{5^2\pi}{3(14)^2}}_{c} h^3$$

$$\boxed{\frac{dV}{dt} = c\, 3h^2 \frac{dh}{dt}}$$

we've related the rates!

$$\frac{dV_{in}}{dt} - 9600 = 3c\, \underbrace{h^2 \frac{dh}{dt}}$$

we know these

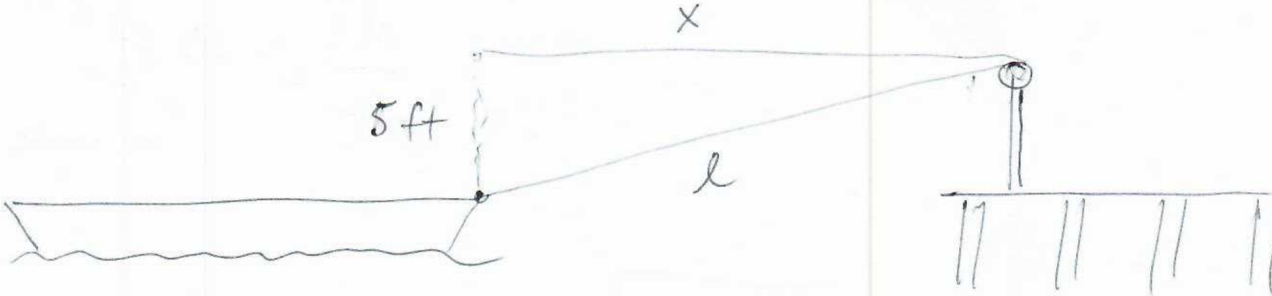
when $h = 1.5\,m = 150\,\text{cm}$

$\frac{dh}{dt} = 22 \ \text{cm}/\text{min}$

$$\frac{dV_{in}}{dt} = 9600 + \frac{5^2\pi}{14^2}(150)^2 \cdot 22$$

$$\simeq 208000 \ \text{cm}^3/\text{min}$$

#4



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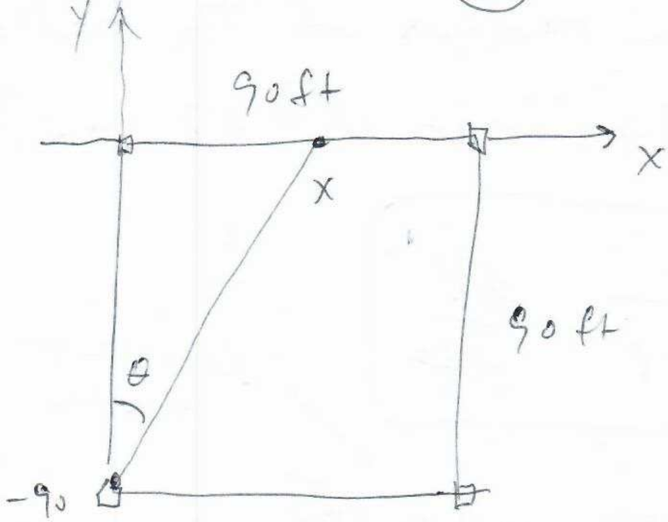
& $\dfrac{dl}{dt} = -2$.

$$\therefore\ 24\frac{dx}{dt} = -4 \cdot 13 \qquad \boxed{\frac{dx}{dt} = \frac{-13}{6} \text{ ft/s}}$$

$\simeq 2.2$ ft/s

3

#6



$\frac{dx}{dt} = -24 \, ft/s$

(x is decreasing, heading for 0).

Relate to $\frac{d\theta}{dt}$

$$\tan\theta = \frac{x}{90}$$

Differentiate both sides!

$$\frac{1}{1+\theta^2}\frac{d\theta}{dt} = \frac{1}{90}\frac{dx}{dt} = \frac{-24}{90} \, \cancel{\frac{dx}{dt}} \, ft/s$$

$$\frac{d\theta}{dt} = \frac{-4}{15}(1+\theta^2)$$

When $x = 30$ ft,

$$\frac{d\theta}{dt} = \frac{-4}{15}\left(1+(\arctan \tfrac{1}{3})^2\right)$$

$$\approx -.294 \text{ radians}/s$$