The Five Lagrange Points: PARKING PLACES IN SPACE

Part I: Finding Lagrange Points

by Dwight E. Neuenschwander, Southern Nazarene University, Emeritus

The James Webb Space Telescope (JWST) is now safely parked at Lagrange Point L2 in the Sun-Earth system, about 1.5 million kilometers from Earth (Fig. 1). A Lagrange point is a location in the vicinity of a gravitationally bound, two-body system where a small object, such as a satellite or an asteroid, maintains a stationary position relative to the two major bodies. Five Lagrange points, called L1 through L5, exist in the vicinity of a gravitationally bound, two-body system. Three of them, L1, L2, and L3, that lie on the line through the two major bodies,[1] were predicted by Leonhard Euler in 1760. With the two major bodies, L4 and L5 form the vertices of equilateral triangles, as predicted by Joseph-Louis Lagrange in 1772 (Fig. 2).[2]

In many readily available accounts of Lagrange points, the results are presented, but few details about the derivation are given. Therefore I have tried to make details of the entire journey accessible here. This article is divided into two parts: I. Finding Lagrange Points, and II. Mechanical Stability at Lagrange Points. Part II will be published in the Spring 2024 issue of *Radiations*.

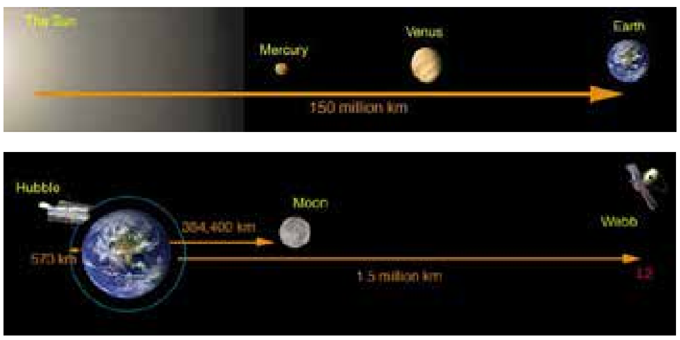

Figure 1. *Top: The distance between the Sun and Earth. Bottom: JWST (the Webb) orbits the Sun 1.5 million kilometers away from the Earth at the second Lagrange point or L2 of the Sun-Earth system. Credit: NASA.*

A Zero-g Location Is Not a Lagrange Point

Every student of introductory physics has probably seen the following problem on a weekly quiz: A star of mass m_1 and its planet of mass m_2 are separated by distance a. Where along the line between them does the gravitational field $\mathbf{g}$ vanish? Let that point be located at distance x from m_1. Then $\mathbf{g}(x)$ between the two masses is

$$\mathbf{g}(x) = \left(-\frac{Gm_1}{x^2} + \frac{Gm_2}{(a-x)^2}\right)\hat{\mathbf{\imath}}. \qquad (1)$$

Setting $\mathbf{g} = 0$ and solving for x gives

$$x = \frac{a}{1+\sqrt{\frac{m_2}{m_1}}}. \qquad (2)$$

As a consistency check, if $m_1 = m_2$ then Eq. (2) gives $x = a/2$, as expected. The x of Eq. (2) does not locate a Lagrange point, because this elementary exercise models a *static* star-planet system. But in a real two-body gravitationally bound system, m_1 and m_2 must revolve around their center of mass (CM) in order to avoid falling directly into one another. In the Sun-Earth system the CM resides about 450 km from the Sun's center, well below its surface radius of 6.7×10^5 km, whereas the two bodies are about 150 million km apart. As the Earth revolves around the Sun, the Sun's center wobbles about their CM. In the two-body problem we must handle this recoil effect.

Before leaving the elementary exercise, let's see how to reformulate the problem in terms of the gravitational potential φ, where $\mathbf{g} = -\nabla\varphi$. For the system of Eq. (1),

$$\varphi(x) = -\frac{Gm_1}{x} - \frac{Gm_2}{a-x}, \qquad (3)$$

and Eq. (2) follows by setting $d\varphi/dx = 0$. Is the zero-$\mathbf{g}$ point of Eq. (2) a point of stable equilibrium, or is it unstable? The second derivative of $\varphi(x)$ is

$$\frac{d^2\varphi}{dx^2} = -2G\left(\frac{m_1}{x^3} + \frac{m_2}{(a-x)^3}\right), \qquad (4)$$

which is negative at the value of x given by Eq. (2), showing x to be a local *maximum* of the potential. A test particle sitting at x will be in a state of unstable mechanical equilibrium.

While the x of Eq. (2) is not a Lagrange point, the same procedure—with the revolution of the bodies about their CM included—will be used to find the Lagrange points.

Relative Motion and the Reduced Mass

Consider two particles, one of mass m_1 located at position $\mathbf{r}_1$ relative to the origin of an arbitrary coordinate system, and the other body a mass m_2 located at position $\mathbf{r}_2$ relative to the same origin. The translational kinetic energy K of this two-particle system is

$$K = \frac{1}{2}m_1\mathbf{v}_1^2 + \frac{1}{2}m_2\mathbf{v}_2^2\,, \tag{5}$$

where $\mathbf{v}_1 = d\mathbf{r}_1/dt$ and $\mathbf{v}_2 = d\mathbf{r}_2/dt$. The CM is located at $\mathbf{r}_{\text{CM}}$, given by

$$\mathbf{r}_{\text{CM}} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}\,. \tag{6}$$

Let **r** denote the vector from m_1 to m_2, so that

$$\mathbf{r} = \mathbf{r_2} - \mathbf{r_1}\,. \tag{7}$$

Use Eqs. (5) and (6) to write $\mathbf{r}_1$ and $\mathbf{r}_2$, and thus $\mathbf{v}_1$ and $\mathbf{v}_2$, in terms of two other velocities: the CM velocity $\mathbf{v}_{\text{CM}} = d\mathbf{r}_{\text{CM}}/dt$ and the relative velocity $\mathbf{v} = d\mathbf{r}/dt = \mathbf{v_2} - \mathbf{v_1}$. In terms of these velocities, the kinetic energy of Eq. (5) is equal to

$$K = \frac{1}{2}(m_1 + m_2)\mathbf{v}_{\text{CM}}^2 + \frac{1}{2}\mu\mathbf{v}^2, \tag{8}$$

where μ denotes the so-called "reduced mass,"

$$\mu \equiv \frac{m_1 m_2}{m_1 + m_2}\,. \tag{9}$$

For a two-body interaction where the force $\mathbf{F}$ between the bodies depends only on the distance between them, so that $\mathbf{F}(\mathbf{r}) = \mathbf{F}(r)$, where $r = |\mathbf{r}|$ (a "central" force), the two-body problem is equivalent to two one-body problems: the motion of the CM (as a particle of mass $m_1 + m_2$) relative to the original origin of coordinates, and the motion of a particle of mass μ acted on by $\mathbf{F}(r)$. The distance r is also the distance from the CM to the effective mass μ. Placing the CM at rest on the origin further reduces this two-body problem to a one-body problem. The motion of the effective particle of mass μ is solved by applying Newton's second law:

$$\mathbf{F}(r) = \mu\frac{d^2\mathbf{r}}{dt^2}\,. \tag{10}$$

Consider the reduced mass effect on the orbital angular frequency in Kepler's third law. But first, for the sake of comparison, consider a central body of mass m_1 orbited by a satellite of mass m_2 where $m_2 \ll m_1$ so that m_1 does not measurably recoil in reaction to the motion of m_2. The CM effectively resides at the center of m_1. Assuming a circular orbit of m_2 about m_1, Newton's second law becomes

$$\frac{Gm_1m_2}{r^2} = m_2\frac{v^2}{r}\,, \tag{11}$$

where v may be expressed in terms of the orbital angular velocity ω, $v = r\omega$. Equation (11) becomes the simplest form of Kepler's third law,

$$\omega^2 = \frac{Gm_1}{r^3}\,. \tag{12}$$

Now let's see how Kepler's third law gets modified with the recoil of m_1 taken into account. With the origin still coincident with the CM, Eq. (10) gives for a circular orbit of the reduced mass

$$\frac{Gm_1m_2}{r^2} = \mu\frac{v^2}{r}\,. \tag{13}$$

With $v = r\omega$, Eq. (13) becomes

$$\omega^2 = \frac{G(m_1 + m_2)}{r^3}\,, \tag{14}$$

which reduces to Eq. (12) if $m_2 \ll m_1$. Equation (14) gives the orbital frequency that applies to the Lagrange points.

$\mathbf{F} = m\mathbf{a}$ Transformed into a Rotating Reference Frame

According to Newton's first law, physics is done most efficiently when the second law is applied in an unaccelerated (inertial) reference frame where the distinction between cause (force) and effect (acceleration) is unambiguous. To do physics in a rotating frame, start with Newton's laws in an inertial frame, then carry out a coordinate transformation from the inertial to the rotating frame. Let the rotating frame rotate with constant angular velocity $\boldsymbol{\omega}$ with respect to the inertial frame (and assume no relative rectilinear acceleration). Applied to a particle of mass m', the transformation from an inertial to the rotating frame results in a modified second law[3]:

$$\mathbf{F} - m'\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) - 2m'(\boldsymbol{\omega} \times \mathbf{v}) = m'\mathbf{a}\,, \qquad (15)$$

where **r**, **v**, and **a** are the position, velocity, and acceleration of m' as measured *within the rotating frame*, and $\mathbf{F}$ is the net force acting on the particle back in the inertial frame. The other two terms that appear with $\mathbf{F}$ are artifacts of doing physics in a rotating reference frame: $-m'\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$ is called the centrifugal "force" and $-2m'(\boldsymbol{\omega} \times \mathbf{v})$ the Coriolis "force."

Corresponding to a conservative force $\mathbf{F}$ is potential energy U, where

$$U = -\int \mathbf{F} \cdot d\mathbf{r}. \qquad (16)$$

An effective potential energy U_ω can likewise be defined for the force $\mathbf{F}_\omega \equiv \mathbf{F} - m'\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) - 2m'(\boldsymbol{\omega} \times \mathbf{v})$:

$$U_\omega \equiv -\int \boldsymbol{F_\omega} \cdot d\mathbf{r}$$

so that

$$U_\omega = U + m'\boldsymbol{\omega} \times \left(\boldsymbol{\omega} \times \int \mathbf{r} \cdot d\mathbf{r}\right) + 2m' \int (\boldsymbol{\omega} \times \mathbf{v}) \cdot d\mathbf{r}. \qquad (17)$$

A potential energy corresponding to the Coriolis force does not exist because $(\boldsymbol{\omega} \times \mathbf{v}) \cdot d\mathbf{r} = (\boldsymbol{\omega} \times \mathbf{v}) \cdot \mathbf{v}\, dt = 0$.[4] Even so, for now consider the test particle to be at rest relative to the rotating frame, so that $\mathbf{v} = \mathbf{0}$ (a restriction lifted later). In the plane polar coordinate system (r, θ) with the CM at the origin, and with $\boldsymbol{\omega} = \omega\hat{\mathbf{k}}$, Eq. (17) becomes

$$U_\omega = U - \frac{1}{2}m'r^2\omega^2. \qquad (18)$$

Upon dividing out m' we obtain the effective potential φ_ω,

$$\varphi_\omega = \varphi - \frac{1}{2}r^2\omega^2, \qquad (19)$$

where $\varphi = -\int \mathbf{g} \cdot d\mathbf{r}$.

To find the equilibrium points that a test particle experiences in this rotating frame, we set $\mathbf{g}_{\boldsymbol{\omega}} = -\boldsymbol{\nabla}\varphi_\omega = \mathbf{0}$. In plane polar coordinates this requires

$$\boldsymbol{\nabla}\varphi_\omega = \hat{\mathbf{r}}\,\frac{\partial\varphi_\omega}{\partial r} + \hat{\boldsymbol{\theta}}\,\frac{1}{r}\frac{\partial\varphi_\omega}{\partial\theta} = \mathbf{0}\,. \qquad (20)$$

Again, let m_1 and m_2 denote the masses of the two principal bodies, and suppose that $m_1 \geq m_2$. Let the origin again coincide with the CM, and denote r_1 and r_2, respectively, as the distances of m_1 and m_2 from the CM, with a the distance between them (Fig. 2). Then

$$m_1r_1 = m_2r_2 \qquad (21)$$

and

$$a = r_1 + r_2\,, \qquad (22)$$

and denote

$$M = m_1 + m_2\,. \qquad (23)$$

At an arbitrary point in the (r, θ) plane, the effective gravitational potential is

$$\varphi_\omega(r, \theta) = -\frac{Gm_1}{s_1} - \frac{Gm_2}{s_2} - \frac{1}{2}r^2\omega^2, \qquad (24)$$

where, by the law of cosines (Fig. 2),

$$s_1 = (r^2 + r_1^2 + 2rr_1\cos\theta)^{1/2} \qquad (25)$$

and

$$s_2 = (r^2 + r_2^2 - 2rr_2\cos\theta)^{1/2}. \qquad (26)$$

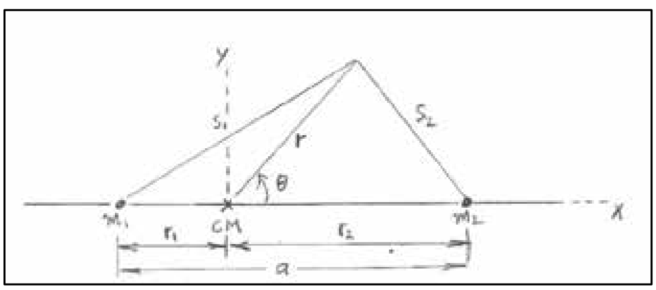

Figure 2. *The coordinate system used to find Lagrange points.*

To locate equilibrium points we require $\nabla\varphi_\omega = \mathbf{0}$. For $\partial\varphi_\omega/\partial r = 0$, and using Eq. (14) to write ω^2 in terms of G, we obtain

$$\frac{\partial\varphi_\omega}{\partial r} = \frac{Gm_1}{s_1^3}(r + r_1\cos\theta) + \frac{Gm_2}{s_2^3}(r - r_2\cos\theta) - r\frac{GM}{a^3} = 0\,. \qquad (27)$$

With $m_1 r_1 = m_2 r_2$ this may be rearranged as

$$r\left(\frac{m_1}{s_1^3} + \frac{m_2}{s_2^3}\right) + m_1 r_1\cos\theta\left(\frac{1}{s_1^3} - \frac{1}{s_2^3}\right) - \frac{rM}{a^3} = 0\,. \qquad (28)$$

Turning to $\partial\varphi_\omega/\partial\theta = 0$, and again using $m_1 r_1 = m_2 r_2$, we find

$$\sin\theta\left(\frac{1}{s_1^3} - \frac{1}{s_2^3}\right) = 0\,. \qquad (29)$$

Equation (29) presents us with two scenarios, depending on whether $\sin\theta$ does not or does vanish.

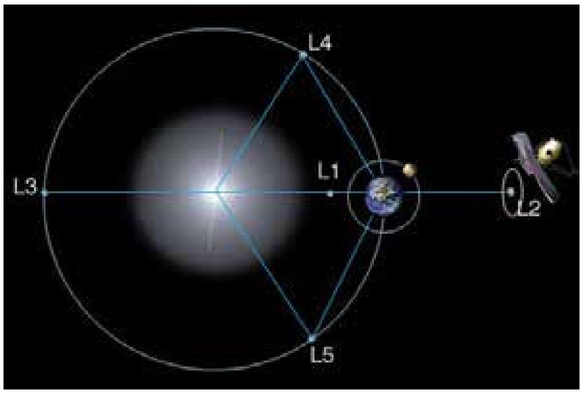

Figure 3. Lagrange points for the Earth-Sun system. L4 and L5 lie at vertices of two symmetrically placed equilateral triangles whose sides have length a. L1 and L2 are about 1.5×10^6 km (0.01 AU) from Earth; L3 is about 1 AU from the Sun. JWST is at L2. Credit: NASA.

$\sin\theta \neq 0$ Produces L4 and L5

If $\sin\theta \neq 0$, then Eq. (29) requires $s_1 = s_2$. Importing this into Eq. (27) immediately gives $s_1 = s_2 = a$. Because $\cos(-\theta) = \cos\theta$, it follows that two distinct Lagrange points exist off the m_1- m_2 axis, symmetrically placed about that axis. Together with m_1 and m_2, each of these Lagrange points forms the third vertex of an equilateral triangle having sides of length a. The Lagrange point above the *x*-axis (ahead of the planet's rotation about the CM) in Fig. 3 is called L4, and the one symmetrically placed below the *x*-axis (following the planet's rotation) is L5.

With the *xy*-coordinate system of Fig. 2, the *x*-coordinate of L4 and L5 is $x = a\cos 60^\circ = a/2$, and the y-components of L4 and L5 are $y = \pm a\sin 60^\circ = \pm\sqrt{3}a/2$, plus for L4 and minus for L5. From Eqs. (21)–(23) we learn that

$$r_1 = \frac{m_2}{M}a \equiv \alpha a \qquad (30)$$

and

$$r_2 = \frac{m_1}{M}a \equiv \beta a. \qquad (31)$$

$\sin\theta = 0$ Produces L1, L2, and L3

If $\sin\theta = 0$ then $\cos\theta = \pm 1$, which means that any remaining Lagrange points lie somewhere along the x-axis. Because the *x*-coordinate will be positive to the right of the origin and negative to its left, I'll write the distances in this section using the radial coordinate *r*, which is non-negative.

We encounter three possibilities for Lagrange points on the *x*-axis: (a) L1 denotes the Lagrange point between m_1 and m_2; (b) L2 stands to the right of m_2, where $r > r_2$; and (c) L3 sits to the left of m_1, where $r > r_1$. Let's consider each of these separately.

(a) L1: From Eq. (27),

$$\varphi_\omega = -\frac{Gm_1}{r + r_1} - \frac{Gm_2}{r_2 - r} - \frac{Gr^2 M}{2a^3}\,. \qquad (32)$$

Setting $\partial\varphi_\omega/\partial r = 0$ and recalling Eqs. (30) and (31) gives

$$\frac{\beta}{(r + r_1)^2} - \frac{\alpha}{(r_2 - r)^2} = \frac{r}{a^3}\,. \qquad (33)$$

(b) L2, $r > r_2$:

$$\varphi_\omega = -\frac{Gm_1}{r+r_1} - \frac{Gm_2}{r-r_2} - \frac{Gr^2M}{2a^3}. \quad (34)$$

Setting $\partial\varphi_\omega/\partial r = 0$, we find

$$\frac{\beta}{(r+r_1)^2} + \frac{\alpha}{(r-r_2)^2} = \frac{r}{a^3}. \quad (35)$$

(c) L3, $r > r_1$:

$$\varphi_\omega = -\frac{Gm_1}{r-r_1} - \frac{Gm_2}{r+r_2} - \frac{Gr^2M}{2a^3}. \quad (36)$$

Setting $\partial\varphi_\omega/\partial r = 0$ results in

$$\frac{\beta}{(r-r_1)^2} + \frac{\alpha}{(r+r_2)^2} = \frac{r}{a^3}. \quad (37)$$

Equations (33), (35), and (37) are to be solved for their respective values of *r*. These are fifth-order polynomials in *r*, so high-precision solutions require numerical methods. However, with some approximations we can derive analytic estimates for the three values of *r* that locate L1, L2, and L3. If $m_2 \ll m_1$, then from Eqs. (30) and (31) we may say $\alpha \ll 1$ and $\beta \approx 1$. Returning to Eq. (33) for L1, it becomes

$$\frac{1}{(r+r_1)^2} - \frac{\alpha}{(r_2-r)^2} \approx \frac{r}{a^3}. \quad (38)$$

With the change of variable $r = au + r_2 = a(u+\beta) \approx a(u+1)$, and recalling $a = r_1 + r_2$, Eq. (38) becomes

$$\frac{1}{(1+u)^2} - \frac{\alpha}{u^2} \approx u + 1. \quad (39)$$

Since $\alpha \ll 1$, it follows that $r \approx r_2$ so that $u \ll 1$ also. In Eq. (39) the binomial expansion of $(1+u)^{-2}$ gives to first order in u the result $u \approx -\sqrt[3]{\alpha/3}$, and thus $r \approx a(1+u) \approx a(1-\sqrt[3]{\alpha/3})$. With the same change of variable for L2 of Eq. (35), and with a similar treatment for L3 on Eq. (37) (but for L3 set $r = au + r_1$), the approximate solutions can be neatly summarized[5]:

$$\text{L1:} \quad r \approx a\left[1 - \left(\frac{\alpha}{3}\right)^{1/3}\right] \quad (40a)$$

$$\text{L2:} \quad r \approx a\left[1 + \left(\frac{\alpha}{3}\right)^{1/3}\right] \quad (40b)$$

$$\text{L3:} \quad r \approx a\left[1 + \frac{\alpha}{3}\right]. \quad (40c)$$

Notice that L1 and L2 are symmetrically placed along the *x*-axis about m_2.

Lagrange points exist at the star-planet and planet-satellite scales. Consider the Sun-Earth system. The Sun's mass $m_1 \approx 2 \times 10^{30}$ kg, for Earth $m_2 \approx 6 \times 10^{24}$ kg, so that $\alpha/3 = m_2/3M \approx 1 \times 10^{-6}$ and $\sqrt[3]{\alpha/3} \approx 1 \times 10^{-2}$. The Sun-Earth center-to-center separation *a* = 1 AU is about 150 million km or 93 million miles. From Eqs. (40) our numbers place L1 at 0.99*a* and L2 at 1.01*a*. This puts L1 and L2 at about 1% of an astronomical unit, or 1.5 million kilometers, equal to 930,000 miles from Earth, with L1 between Earth and the Sun and L2 about 690,000 miles beyond the Moon's orbit radius. In our approximation L3 sits about 93 million plus 93 additional miles, or 1.497×10^6 km on the far side of the Sun, directly opposite the Earth, which compares favorably to the tabulated value of 149.6×10^6 km.[6] L4 and L5 are 1 AU from the Sun on the same orbital path as Earth, L4 60 degrees ahead and L5 60 degrees behind the Earth.

In part II, to appear in the next issue of *Radiations*, we will discuss mechanical stability at Lagrange points.

References

1. I am using the numbering conventions of the space community (NASA, etc.) for labeling the Lagrange points. Some astronomers interchange the labels for L1 and L2.

2. Joseph-Louis Lagrange, *Essai sur le Problème des Trois Corps* (1772).

3. Jerry B. Marion, *Classical Dynamics of Particles and Systems*, 2nd ed. (New York: Academic Press, 1970), Ch. 11. See also D. E. Neuenschwander, "When **F** Does Not Equal *m***a**," *SPS Observer*, Spring/Summer 2001, 10–13.

4. The reader will recall that the magnetic force $q(\mathbf{v} \times \mathbf{B})$ does no work and therefore has no scalar potential energy function in general, because $q(\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{r} = q(\mathbf{v} \times \mathbf{B}) \cdot \mathbf{v}\, dt = 0$. Magnetic forces and the Coriolis force *steer* moving particles but do not change their kinetic energies.

5. See Neil J. Cornish, "The Lagrange Points," an online document created for WMAP Education and Outreach/1998, available at map.gsfc.nasa.gov/ContentMedia/lagrange.pdf. For the *r* of L3 I get $r = a(1 + 4\alpha/12)$, whereas Cornish obtains $r = a(1 + 5\alpha/12)$. Both are approximations.

6. Available at en.wikipedia.org/wiki/Lagrange_point.