Chapter2 Worksheet

Power, Quotient, Exponential, and Sine/Cosine Worksheet

1. Let $f(x) = \sqrt{x}$.

1.1. Get the linear approximation to $f(x)$ near $x = 1$. Call it $L(x)$.

In[144]:=
```
f[x_] := Sqrt[x]
lf[x_] = f[1] + f'[1] (x - 1)
```

Out[145]= $1 + \frac{1}{2}(-1 + x)$

1.2. Make a table of values for your linear approximations, the exact values, and the absolute errors.

x	−0.2	0.1	0.4	0.7	1.0	1.3	1.6	1.9	2.2	2.5
$L(x)$										
$f(x)$										
$\lvert f(x) - L(x)\rvert$										

Here's Mathematica's version of that: I'm going to transpose it, so that it will show up in the pdf:

In[146]:=
```
MatrixForm[Table[{x, lf[x], f[x], Abs[f[x] - lf[x]]}, {x, -0.2, 2.5, 0.3}]]
```

Out[146]//MatrixForm=

-0.2	0.4	0. + 0.447213595499958 i	0.6
0.1	0.55	0.316227766016838	0.233772233983162
0.4	0.7	0.632455532033676	0.0675444679663241
0.7	0.85	0.836660026534076	0.0133399734659244
1.	1.	1.	0.
1.3	1.15	1.14017542509914	0.00982457490086186
1.6	1.3	1.26491106406735	0.0350889359326481
1.9	1.45	1.37840487520902	0.0715951247909779
2.2	1.6	1.48323969741913	0.116760302580867
2.5	1.75	1.58113883008419	0.16886116991581

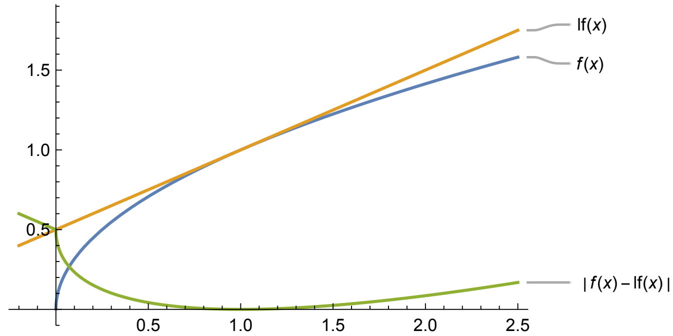
1.3. For which values of x does your linear approximation fall within 0.1 in absolute error of the exact value?

0.4, 0.7, 1, 1.3, 1.6, 1.9: those values closest to 1. Farther away, it falls off. By the way, notice that f(0.2) is imaginary! But we can still compute the absolute value!

If we have a look at the graphs, we see clearly how well the local linear approximation works around 1, and how it breaks down further away. And even though the square root isn't defined for the negative real line, the local linearization is! So it's giving "ridiculous" results there....

In[147]:=
```
Plot[{f[x], lf[x], Abs[f[x] - lf[x]]}, {x, -.2, 2.5}, PlotLabels → Automatic]
```

Out[147]=



2. Let $h(x) = \frac{5x+5}{x^2+x+1}$.

In[148]:=
```
h[x_] := (5 x + 5) / (x^2 + x + 1)
```

2.1. Is $h'(5)$ negative, 0, or positive? Write the equation of the tangent line at $x = 5$, **in point-slope form** (do not "simplify").

In[149]:=
```
Simplify[h'[x]]
h'[5]
lh[x_] = h[5] + h'[5] (x - 5)
```

Out[149]= $-\frac{5\,x\,(2+x)}{\left(1+x+x^2\right)^2}$

Out[150]= $-\frac{175}{961}$

Out[151]= $\frac{30}{31} - \frac{175}{961}(-5+x)$

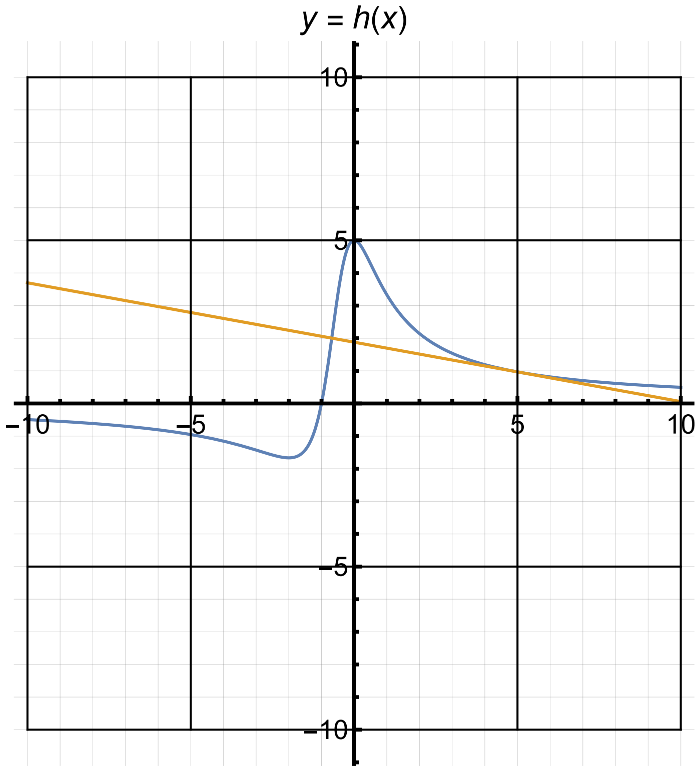
So the derivative is negative at 5.

2.2. Determine all the values of x where the $y = h(x)$ will have horizontal tangents by solving $h'(x) = 0$.

From the formula above, $-\frac{5\,x\,(2+x)}{\left(1+x+x^2\right)^2}$, h'(x) is 0 when x=0 and when x=-2.

2.3. Graph $y = h(x)$ below. Put points on the graph where $x = 5$ and where you found horizontal tangents. Does the graph match what you expected in the questions above?

Out[152]=



3. Consider the function $f(x) = 4\sin(x)$. Its graph is shown below.

In[153]:=
```
f[x_] := 4 Sin[x]
tl[x_, x0_] := f[x0] + f'[x0] (x - x0)
```

3.1. Draw the tangent line at each of the give x-values and estimate its slope. (Get the intermediate values by symmetry!)

x	−6	□	−4	□	−2	□	0	1	□	3	□
slope of tangent line											

Here is Mathematica’s version of that table: I’m going to transpose it, so that it will show up in the pdf:

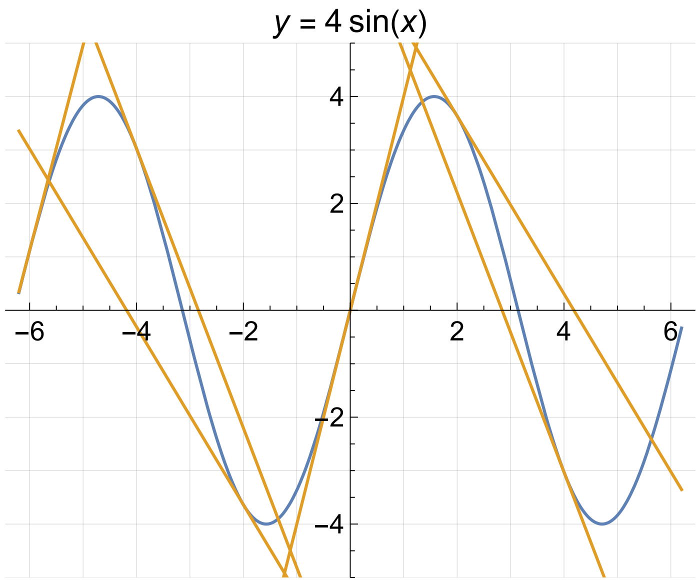
In[155]:=
```
MatrixForm[Table[{x, f'[x]}, {x, -6, 6, 1.0}]]
```

Out[155]//MatrixForm=

-6.	3.84068114660146
-5.	1.13464874185291
-4.	-2.61457448345445
-3.	-3.95996998640178
-2.	-1.66458734618857
-1.	2.16120922347256
0.	4.
1.	2.16120922347256
2.	-1.66458734618857
3.	-3.95996998640178
4.	-2.61457448345445
5.	1.13464874185291
6.	3.84068114660146

In[156]:=
```
Plot[{f[x], Table[tl[x, xo], {xo, -6, 5, 2}]}, {x, -6.2, 6.2}, PlotRange → {-5, 5},
 AspectRatio → Automatic, GridLines → {Range[-6, 6], Range[-6, 6]},
 PlotLabel → HoldForm[y = 4 Sin[x]], BaseStyle → FontSize → 14]
```

Out[156]=



3.2. Compute the derivative $f'(x)$ and use it to make the table of values.

x	−6	−5	−4	−3	−2	−1	0	1	2	3	4	5	6
$f'(x)$													

3.3. Did the two tables match?

Mine did, by design!:)

4. Let $h(x) = \frac{\sin(x)}{1+x^2}$.

In[157]:=
```
h[x_] := Sin[x] / (1 + x^2)
```

4.1. Determine the linear approximation $L(x)$ to $h(x)$ around $x = 0$.

In[158]:=
```
Simplify[h'[x]]
h'[0]
lh[x_] = h[0] + h'[0] (x - 0)
```

Out[158]= $\frac{(1+x^2)\,\text{Cos}[x] - 2\,x\,\text{Sin}[x]}{(1+x^2)^2}$

Out[159]= 1

Out[160]= x

That's a pretty simple linearization!

4.2. Use your linear approximation to estimate $h(-0.2)$.

In[161]:=
```
h[-0.2]
lh[-0.2]
```

Out[161]= -0.191028202687559

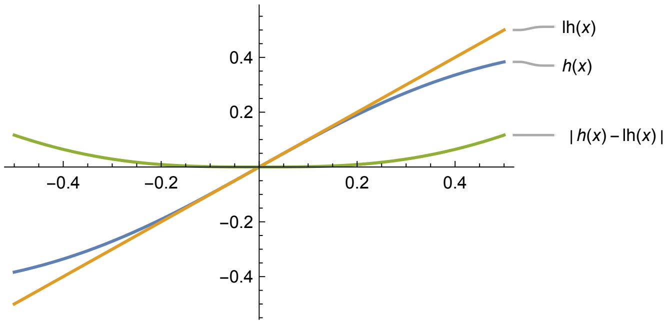
Out[162]= -0.2

Pretty close!

4.3. Plot $L(x)$ and $h(x)$ in the vicinity of $x = 0$, and comment on how well it did at estimating $h(-0.2)$.

It starts to break down about 0.2 units away from 0:

In[163]:=
```
Plot[{h[x], lh[x], Abs[h[x] - lh[x]]}, {x, -.5, .5}, PlotLabels → Automatic]
```

Out[163]=


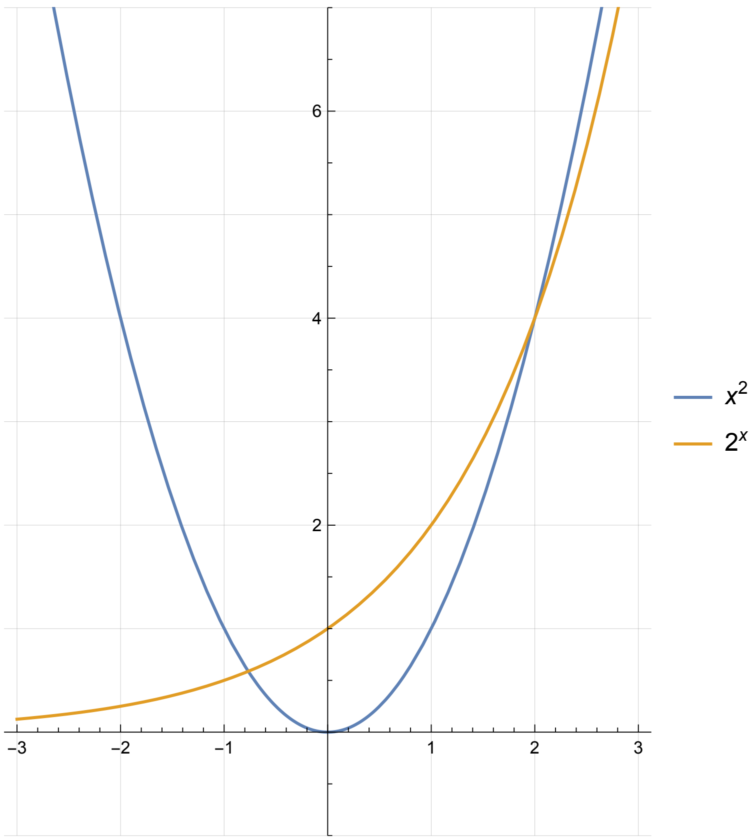
Derivative of exponentials

Comparison

Power function: $f(x) = x^2$

Exponential function: $g(x) = 2^x$

Out[164]=



Derivatives of exponential functions.

- Let $f(x) = 2^x$. Then

$$f'(x) = \lim_{h\to 0} \frac{2^{x+h} - 2^x}{h}$$
$$= \lim_{h\to 0} 2^x \frac{(2^h - 1)}{h}$$
$$= 2^x \lim_{h\to 0} \frac{2^h - 1}{h}$$

h	0.1	0.01	0.001	0.0001	0.00001
$\frac{2^h-1}{h}$	0.717735	0.695555	0.693387	0.693171	0.69315

$f'(x) = 0.693\,(3^x) = 0.693\, f(x)$

- Let $f(x) = 3^x$. Then

$$f'(x) = \lim_{h\to 0} \frac{3^{x+h} - 3^x}{h}$$
$$= \lim_{h\to 0} 3^x \frac{(3^h - 1)}{h}$$
$$= 3^x \lim_{h\to 0} \frac{3^h - 1}{h}$$

h	0.1	0.01	0.001	0.0001	0.00001
$\frac{3^h-1}{h}$	1.16123174033904	1.10466919378536	1.09921598420404	1.09867263832664	1.0986183234251

$$f'(x) = 1.0986\,(3^x) = 1.0986\, f(x)$$

5. Natural base:

We might guess that somewhere between 2 and 3, there is a base such that $f'(x) = f(x)$. In fact, there is a number e with $2 < e < 3$ such that if $f(x) = e^x$, then $f'(x) = 1 \cdot e^x = e^x$.

Euler's number is $e = 2.71828\ldots$ Complete a table like those above, and verify that $f'(x) = f(x)$.

h	0.1	0.01	0.001	0.0001	0.00001
$\frac{e^h-1}{h}$					

```
Out[166]= 2.718281828459045235360287471352662497757247093699959574966967627724076630353547594571382178525166427
```

I'm going to transpose it, so that it will show up in the pdf:

```
In[167]:= MatrixForm[Table[{h, (E^h - 1) / h}, {h, {0.1, 0.01, 0.001, 0.0001}}]]
Out[167]//MatrixForm=
0.1     1.05170918075648
0.01    1.0050167084168
0.001   1.00050016670838
0.0001  1.00005000166714
```

Questions

- 5.1. What is the slope of the tangent line to $y = e^x$ when $x = -1$? When $x = 1$?

  Those are easy, because the slopes are the function values!

```
In[168]:= E^-1
E^1
Out[168]= 1/e
Out[169]= e
```

- 5.2. Find an equation for the tangent line to $y = x^3 - e^x + 2$ when $x = 0$.

In[170]:=
```
f[x_] := x^3 - E^x + 2
lf[x_] = f[0] + f'[0] (x - 0)
```

Out[171]= $1 - x$