ODE example for MAT360

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Tea Time Numerical Analysis, Taylor Methods, starting on p. 209

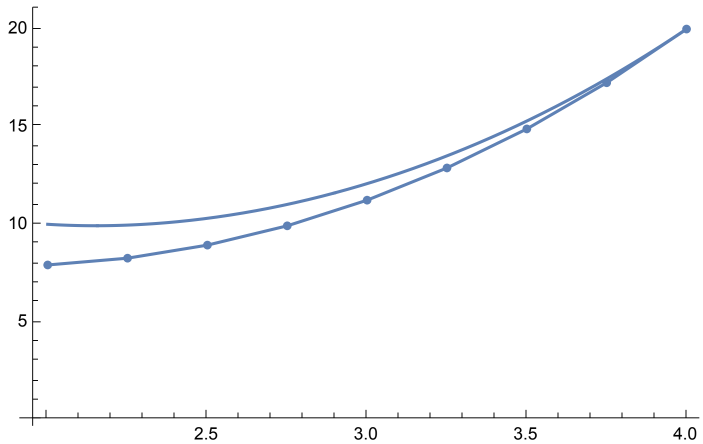
Leave it to Leo to go backwards in time in his first example...:)

In[585]:=
```
euler[f_, y0_, {a_, b_, n_}] :=
 Module[
  (* create some local variables: *)
  {h = N[(b - a) / n], t = a, y = y0, soln},
  (* initialize the solution list *)
  soln = {{t, y}};
  (* iterate, updating y each time: *)
  Do[y = y + h f[t, y];
   (* step time with a constant step-size: *)
   t = t + h;
   (* update soln: *)
   AppendTo[soln, {t, y}
   ],
   (* do this n times: *)
   {n}
  ];
  (* return the solution list of times and y values: *)
  soln
 ]
(* Now we'll recreate Figure 6.2.2, p. 210  *)
f[t_, y_] := -y / t + t^2
y0 = 20;
n = 8;
a = 4;
b = 2;
g[x_] := x^3 / 4 + 16 / x (* exact solution *)

soln = euler[f, y0, {a, b, n}];
MatrixForm[soln]
p0 = ListPlot[soln, Joined → True, InterpolationOrder → 1];
p2 = Plot[g[x], {x, a, b}];
p1 = ListPlot[soln];
Show[p0, p1, p2]
```

Out[593]//MatrixForm=

4	20
3.75	17.25
3.5	14.884375
3.25	12.8850446428571
3.	11.2355769230769
2.75	9.921875
2.5	8.93323863636364
2.25	8.2640625
2.	7.91666666666667

Out[597]=

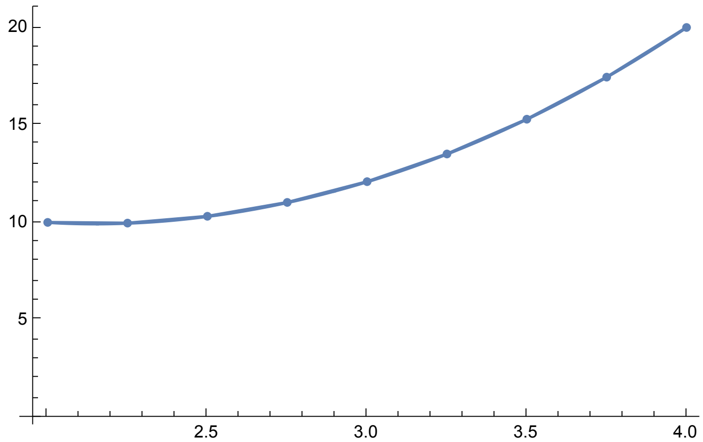


In[598]:=
```
taylor2[f_, fp_, y0_, {a_, b_, n_}] :=
 Module[
  (* create some local variables: *)
  {h = N[(b - a) / n], t = a, y = y0, soln},
  (* initialize the solution list *)
  soln = {{t, y}};
  (* iterate, updating y each time: *)
  Do[y = y + h ( f[t, y] + h / 2 * fp[t, y]);
   (* step time with a constant step-size: *)
   t = t + h;
   (* update soln: *)
   AppendTo[soln, {t, y}
   ],
   (* do this n times: *)
   {n}
  ];
  (* return the solution list of times and y values: *)
  soln
 ]

f[t_, y_] := -y / t + t^2
(* This calculation is done on page 209 *)
fp[t_, y_] := 2 y / t^2 + t
y0 = 20;
n = 8;
a = 4;
b = 2;
g[x_] := x^3 / 4 + 16 / x (* exact solution *)

soln = taylor2[f, fp, y0, {a, b, n}];
MatrixForm[soln]
p0 = ListPlot[soln, Joined → True, InterpolationOrder → 1];
p2 = Plot[g[x], {x, a, b}];
p1 = ListPlot[soln];
Show[p0, p1, p2]
```

Out[607]//MatrixForm=

4	20
3.75	17.453125
3.5	15.2957986111111
3.25	13.5132704435941
3.	12.0936504655486
2.75	11.0291883547996
2.5	10.3183046585813
2.25	9.96894317102529
2.	10.0043643958432

Out[611]=

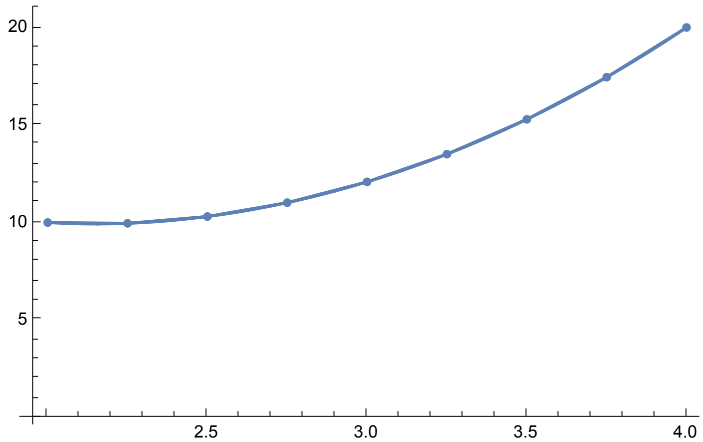


In[612]:=
```
taylor3[f_, fp_, fpp_, y0_, {a_, b_, n_}] :=
 Module[
  (* create some local variables: *)
  {h = N[(b - a) / n], t = a, y = y0, soln},
  (* initialize the solution list *)
  soln = {{t, y}};
  (* iterate, updating y each time: *)
  Do[y = y + h ( f[t, y] + h / 2 * (fp[t, y] + h / 3 fpp[t, y]));
   (* step time with a constant step-size: *)
   t = t + h;
   (* update soln: *)
   AppendTo[soln, {t, y}
   ],
   (* do this n times: *)
   {n}
  ];
  (* return the solution list of times and y values: *)
  soln
 ]
(* Now we'll recreate Table 6.2, p. 212 *)
f[t_, y_] := -y / t + t^2
fp[t_, y_] := 2 y / t^2 + t
(*  I did the following calculation, and hope that it's right...:) *)
fpp[t_, y_] := 2 (t^3 - 3 y) / t^3 + 1
y0 = 20;
n = 8;
a = 4;
b = 2;
g[x_] := x^3 / 4 + 16 / x (* exact solution *)

soln = taylor3[f, fp, fpp, y0, {a, b, n}];
MatrixForm[soln]
p0 = ListPlot[soln, Joined → True, InterpolationOrder → 1];
p2 = Plot[g[x], {x, a, b}];
p1 = ListPlot[soln];
Show[p0, p1, p2]
```

Out[622]//MatrixForm=

4	20
3.75	17.4501953125
3.5	15.2900185185185
3.25	13.5048076611597
3.	12.08282105533
2.75	11.0165611627877
2.5	10.3048895885208
2.25	9.9565448328466
2.	9.99628071047217

Out[626]=



#1-3, ac

In[627]:=
```
f[t_, y_] := 3 t - 2 y
fp[t_, y_] := 3 - 2 (3 t - 2 y)
(*  I did the following calculation, and hope that it's right...:)
     Notice that I cheat and don't simplify y', leaving it as f[t,y]. *)
fpp[t_, y_] := -6 + 4 f[t, y]
y0 = 1;
n = 2;
a = 1;
b = 2;
soln = euler[f, y0, {a, b, n}];
MatrixForm[soln]
soln = taylor2[f, fp, y0, {a, b, n}];
MatrixForm[soln]
soln = taylor3[f, fp, fpp, y0, {a, b, n}];
MatrixForm[soln]
```

Out[635]//MatrixForm=

$$\begin{pmatrix} 1 & 1 \\ 1.5 & 1.5 \\ 2. & 2.25 \end{pmatrix}$$

Out[637]//MatrixForm=

$$\begin{pmatrix} 1 & 1 \\ 1.5 & 1.625 \\ 2. & 2.3125 \end{pmatrix}$$

Out[639]//MatrixForm=

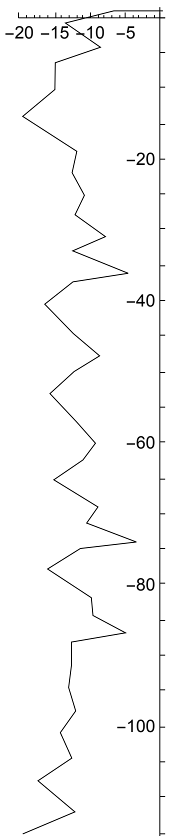
$$\begin{pmatrix} 1 & 1 \\ 1.5 & 1.58333333333333 \\ 2. & 2.27777777777778 \end{pmatrix}$$

It’s easy to operate on systems:

Here’s the pendulum problem:

In[640]:=
```
Clear[x, y, z, f, c, m, g, l]
(* We're thinking of f as a vector-valued function,
obtained by reducing the second order equation for the pendulum to a
  first order system. The arguments a time t and a location (u,v): *)
f[t_, {u_, v_}] := {
  -c / m u - g / l Sin[v],
  u
 }
g = 9.81; (* m/s2 *)
c = 1.1;
l = .31; (* m *)
m = 70; (* kg *)
theta0 = Pi / 3;
thetaprime0 = 0;

(* We just give an initial value that is also a list, of the proper size: *)
soln = euler[f, {thetaprime0, theta0}, {0, 10, 40}];
curve = Table[soln〚k〛〚2〛, {k, 1, Length[soln]}];
Show[Graphics[{Line[curve]}], Axes → Automatic]
```

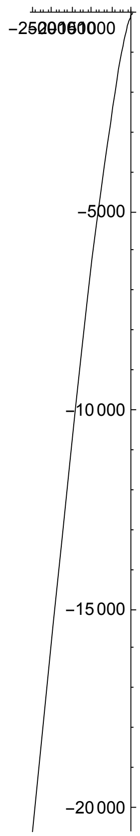
Out[647]=



Let’s try that again, with Taylor 3:

In[648]:=
```
soln = taylor3[f, fp, fpp, {thetaprime0, theta0}, {0, 10, 40}];
curve = Table[soln⟦k⟧⟦2⟧, {k, 1, Length[soln]}];
Show[Graphics[{Line[curve]}], Axes → Automatic]
```

Out[650]=

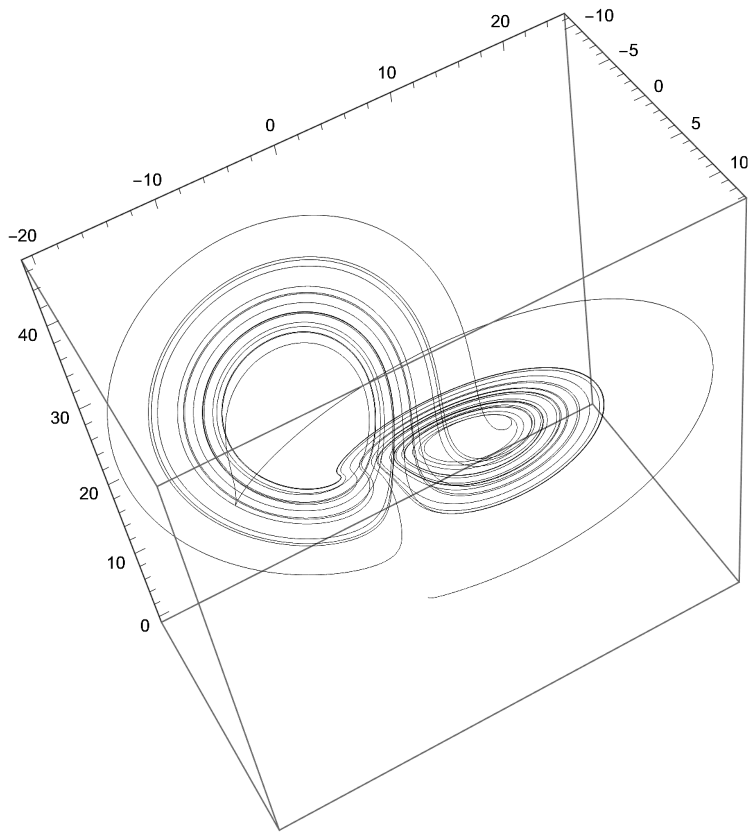


In[651]:=
```
Clear[g]
```

Here is the lovely Lorenz System, exhibiting chaos:

In[652]:=
```
Clear[x, y, z, f]
f[t_, {x_, y_, z_}] := {
  -3. (x - y),
  -x z + 26.5 x - y,
  x y - z
 }
soln = euler[f, {0.0, 1.0, 0.0}, {0, 50, 40 000}];
curve = Table[soln⟦k⟧⟦2⟧, {k, 1, Length[soln]}];
Show[Graphics3D[{Line[curve]}], Axes → Automatic]
```

Out[656]=



In[657]:=

In[658]:=

```
soln = taylor3[f, fp, fpp, {0.0, 1.0, 0.0}, {0, 50, 40000}];
curve = Table[soln[[k]][[2]], {k, 1, Length[soln]}];
Show[Graphics3D[{Line[curve]}], Axes → Automatic]
```

Out[660]=