Chapter 1, Section 2: Homework: pp. 17-19, #4, 7, 12, 14, 24, 27

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4. Suppose $f(x)$ is a function whose fourth derivative exists on the whole real line, $(-\infty, \infty)$, and that $f(2) = 3$, $f'(2) = -1$, $f''(2) = 2$, and $f'''(2) = -1$.

   (a) Write down the third Taylor polynomial for $f(x)$ expanded about $x_0 = 2$.

4 (b) Use the Taylor polynomial to approximate $f(4)$.

   (c) Find a bound on the absolute error of the approximation using the fact that

$$-3 \le f^{(4)}(\xi) \le 5$$

   for all $\xi \in [2, 4]$.

In[87]:=
```
fp4[x_] := Module[{xshift = x - 2},
   3 + xshift ((-1) + xshift / 2 (2 + xshift / 3 (-1)))
  ]
fp4[x]
Together[fp4[x]]
{fp4[2],
 fp4'[2],
 fp4''[2],
 fp4'''[2]}
fp4[4]
Plot[{fp4[x], fp4[x] - 2 (x - 2) ^4 / 4!, fp4[x] + 5 (x - 2) ^4 / 4!},
 {x, 2, 4}, PlotRange → All]
upperBound = 10 / 3 (* 5*2^4/4! *)
```

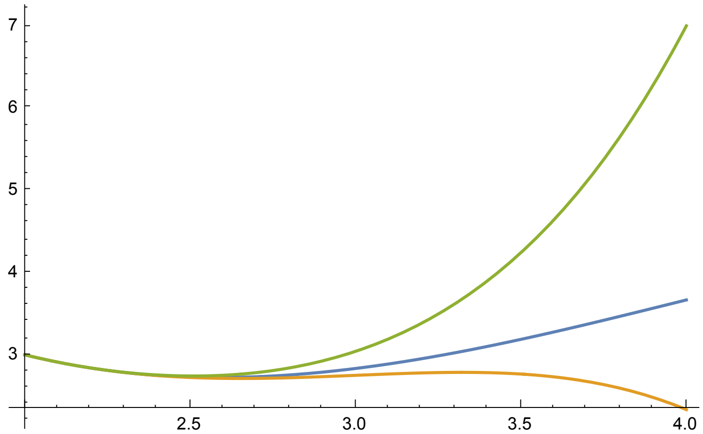
Out[88]= $3 + \left(-1 + \frac{1}{2}\left(2 + \frac{2 - x}{3}\right)(-2 + x)\right)(-2 + x)$

Out[89]= $\frac{1}{6}\left(62 - 42\,x + 12\,x^2 - x^3\right)$

Out[90]= $\{3, -1, 2, -1\}$

Out[91]= $\frac{11}{3}$

Out[92]=



Out[93]= $\frac{10}{3}$

7\.

7\. The hyperbolic sine, $\sinh(x)$, and hyperbolic cosine, $\cosh(x)$, are derivatives of one another. That is,

$$\frac{d}{dx}(\sinh(x)) = \cosh(x)$$

and

$$\frac{d}{dx}(\cosh(x)) = \sinh(x).$$

Find the remainder term, R_{43}, associated with the 43^{rd} Maclaurin polynomial for $f(x) = \cosh(x)$.

Cosh is even, which means the the 43rd is actually of degree 42; but the error term corresponding to the 43rd will be the 44th derivative, corresponding to Cosh(x). So Cosh(ξ(x))x^44/44!.

In[94]:= `Series[Cosh[x], {x, 0, 43}]`

Out[94]=
$$1+\frac{x^2}{2}+\frac{x^4}{24}+\frac{x^6}{720}+\frac{x^8}{40\,320}+\frac{x^{10}}{3\,628\,800}+\frac{x^{12}}{479\,001\,600}+\frac{x^{14}}{87\,178\,291\,200}+$$
$$\frac{x^{16}}{20\,922\,789\,888\,000}+\frac{x^{18}}{6\,402\,373\,705\,728\,000}+\frac{x^{20}}{2\,432\,902\,008\,176\,640\,000}+$$
$$\frac{x^{22}}{1\,124\,000\,727\,777\,607\,680\,000}+\frac{x^{24}}{620\,448\,401\,733\,239\,439\,360\,000}+$$
$$\frac{x^{26}}{403\,291\,461\,126\,605\,635\,584\,000\,000}+\frac{x^{28}}{304\,888\,344\,611\,713\,860\,501\,504\,000\,000}+$$
$$\frac{x^{30}}{265\,252\,859\,812\,191\,058\,636\,308\,480\,000\,000}+\frac{x^{32}}{263\,130\,836\,933\,693\,530\,167\,218\,012\,160\,000\,000}+$$
$$\frac{x^{34}}{295\,232\,799\,039\,604\,140\,847\,618\,609\,643\,520\,000\,000}+$$
$$\frac{x^{36}}{371\,993\,326\,789\,901\,217\,467\,999\,448\,150\,835\,200\,000\,000}+$$
$$\frac{x^{38}}{523\,022\,617\,466\,601\,111\,760\,007\,224\,100\,074\,291\,200\,000\,000}+$$
$$\frac{x^{40}}{815\,915\,283\,247\,897\,734\,345\,611\,269\,596\,115\,894\,272\,000\,000\,000}+$$
$$\frac{x^{42}}{1\,405\,006\,117\,752\,879\,898\,543\,142\,606\,244\,511\,569\,936\,384\,000\,000\,000}+O[x]^{44}$$

---

12. Find $\xi(x)$ as guaranteed by Taylor's theorem in the following situation.

12.

(a) $f(x)=\cos(x)$, $x_0=0$, $n=3$, $x=\pi$. [A]

(b) $f(x)=e^x$, $x_0=0$, $n=3$, $x=\ln 4$.

(c) $f(x)=\ln(x)$, $x_0=1$, $n=4$, $x=2$.

(a)

In[95]:=
```
f[x_] := Cos[x]
tp[x_] = Normal[Series[f[x], {x, 0, 3}]]
fd[x_] := Cos[x]
rp[x_, y_] := fd[y] x^4 / 4!
rp[x, y]
xval = Pi
{y1, y2} = y /. N[Solve[rp[xval, y] == -tp[xval] + f[xval], y]]
ksee = 0.7625353073244038;
f[xval]
tp[xval] + rp[xval, ksee]
```

Out[96]= $1 - \frac{x^2}{2}$

Out[99]= $\frac{1}{24} x^4 \text{Cos}[y]$

Out[100]= π

Out[101]= $\{-0.762535307324404 + 6.28318530717959\, c_1 \text{ if } c_1 \in \mathbb{Z},$
$0.762535307324404 + 6.28318530717959\, c_1 \text{ if } c_1 \in \mathbb{Z}\}$

Out[103]= -1

Out[104]= -1.

In[105]:=
```
N[ArcCos[((12 Pi^2 - 48) / Pi^4)]]
```

Out[105]= 0.762535307324404

(b)

In[106]:=
```
f[x_] := Exp[x]
tp[x_] = Normal[Series[f[x], {x, 0, 3}]]
fd[x_] := Exp[x]
rp[x_, y_] := fd[y] x^4 / 4!
xval = Log[4.0]
ksee = (y /. N[Solve[rp[xval, y] == -tp[xval] + f[xval], y]])[[1]]
f[xval]
tp[xval] + rp[xval, ksee]
```

Out[107]= $1 + x + \frac{x^2}{2} + \frac{x^3}{6}$

Out[110]= 1.38629436111989

Out[111]= 0.304979071848693

Out[112]= 4.

Out[113]= 4.

(c)

```
In[114]:= f[x_] := Log[x]
tp[x_] = Normal[Series[f[x], {x, 1, 4}]]
fd[x_] := Factorial[5] / x^5
rp[x_, y_] := fd[y] (x - 1)^5 / 5!
xval = 2
ksee = (y /. N[Solve[rp[xval, y] == -tp[xval] + f[xval], y]])[[1]]
N[f[xval]]
tp[xval] + rp[xval, ksee]
```

Out[115]= $-1-\frac{1}{2}(-1+x)^2+\frac{1}{3}(-1+x)^3-\frac{1}{4}(-1+x)^4+x$

Out[118]= 2

Out[119]= -1.25842177135699 - 0.914296935058689 ⅈ

Out[120]= 0.693147180559945

Out[121]= 0.693147180559945 + 0. ⅈ

14.

14. Find the second Taylor polynomial, $P_2(x)$, for $f(x) = e^x \cos x$ about $x_0 = 0$.

(a) Use $P_2(0.5)$ to approximate $f(0.5)$. Find an upper bound on the error $|f(0.5) - P_2(0.5)|$ using the remainder term and compare it to the actual error.

(b) Find a bound on the error $|f(x) - P_2(x)|$ good on the interval $[0, 1]$.

(c) Approximate $\int_0^1 f(x)\,dx$ by calculating $\int_0^1 P_2(x)\,dx$ instead.

(d) Find an upper bound for the error in (c) using $\int_0^1 |R_2(x)|\,dx$ and compare the bound to the actual error.

(a) We can multiply the series for e^x and cos(x), to get the series for f(x):

```
Clear[p2]
f[x_] := Exp[x] Cos[x]
p2[x_] = Normal[Series[f[x], {x, 0, 2}]]
Normal[Series[Exp[x], {x, 0, 2}]]
Normal[Series[Cos[x], {x, 0, 2}]]
fppp[x_] = D[f[x], {x, 3}] (* bounded by 2 e^0.5 Sin(0.5) *)
errorBound = 2 E^0.5 (1 + Sin[0.5]) (0.5)^3 / 3!
```

Out[157]= $1 + x$

Out[158]= $1 + x + \frac{x^2}{2}$

Out[159]= $1 - \frac{x^2}{2}$

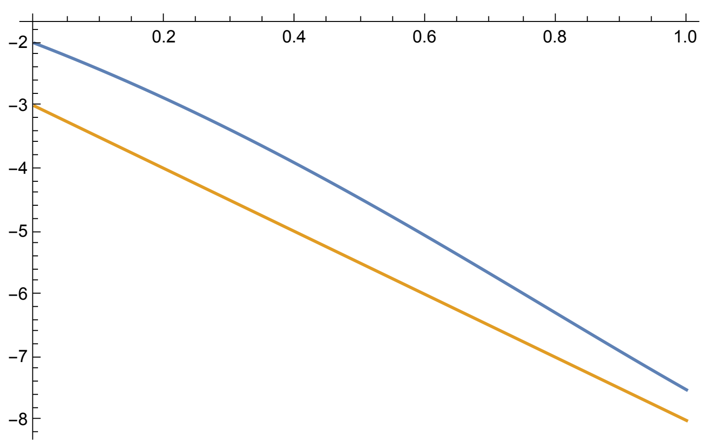
Out[160]= $-2\, e^x \, \text{Cos}[x] - 2\, e^x \, \text{Sin}[x]$

Out[161]= 0.101631681413073

(b) We need the third derivative of f(x), and then to bound it:

In[129]:=
```
fppp[x_] = D[f[x], {x, 3}]
Plot[{fppp[x], -3 - 5 x}, {x, 0, 1}]
M = 8 (* works... *)
```

Out[129]= $-2\, e^x \, \text{Cos}[x] - 2\, e^x \, \text{Sin}[x]$

Out[130]=


Out[131]= 8

more particularly, however, we can create a bound which works for values of x as they slide from 0 to 1: M(x)=3+5x works.

In[132]:=

(c) So let’s compare integrals:

In[133]:=
```
trueVal = N[Integrate[f[x], {x, 0, 1}]]
approxVal = N[Integrate[p2[x], {x, 0, 1}]]
errorVal = Abs[trueVal - approxVal]
```

Out[133]= 1.37802461354736

Out[134]= 1.5

Out[135]= 0.121975386452636

(d) Let’s bound the error:

In[136]:=
```
N[Integrate[(3 + 5 x) * (x^3) / 6, {x, 0, 1}]]
```

Out[136]= 0.291666666666667

24\. 24. Find a bound on the error of the approximation

$$e^2 \approx 1 + 2 + \frac{1}{2}(2)^2 + \frac{1}{6}(2)^3 + \frac{1}{24}(2)^4 + \frac{1}{120}(2)^5$$

according to Taylor’s Theorem. Compare this bound to the actual error.

We’re using the Maclaurin series polynomial to approximate; the error is bounded by e^2 -- the maximum value of the sixth derivative on the interval [0,2] -- multiplied by 2^6/6!

In[137]:=
```
N[E^2 * 2^6 / 6!]
E^2
f[x_] = Normal[Series[Exp[x], {x, 0, 5}]]
f[x]
f[2]
N[Abs[E^2 - f[2]]]
```

Out[137]= 0.656804986571613

Out[138]= e^2

Out[139]= $1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120}$

Out[140]= $1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120}$

Out[141]= $\frac{109}{15}$

Out[142]= 0.122389432263984

27. Let $f(x) = \ln(1 + x)$ and $x_0 = 0$.

(a) Find $T_3(x)$.

(b) Find $R_3(x)$.

(c) Use $T_3(x)$ to approximate $f(1)$ and $f(26)$.

(d) Find a theoretical upper bound on the absolute error of each of the approximations in part (c).

27.

(e) Find a theoretical lower bound on the absolute error of each of the approximations in part (c).

(f) Find the actual absolute error for each of the approximations in part (c). Verify that they are indeed between the theoretical bounds.

(g) Sketch graphs of $f(x)$ and $T_2(x)$ on the same set of axes for $x \in [1, 26]$.

```
f[x_] := Log[1 + x]
(* a *)
t3[x_] = Normal[Series[f[x], {x, 0, 3}]]
(* b *)
D[f[x], {x, 4}]
r3[x_, ksee_] := (-6 / (1 + ksee) ^4) x^4 / 4!
r3[x_] := 6 x^4 / 4! (* but an upper bound on the error is this... *)
(* c *)
N[{f[1], t3[1]}]
N[{f[26], t3[26]}]
(* d *)
{r3[1], r3[26]}
(* e *)
0
(* f *)
N[Abs[f[1] - t3[1]]]
N[Abs[f[26] - t3[26]]]
(* g *)
Plot[{f[x], t3[x]}, {x, 0.01, 2}]
Plot[{f[x], t3[x]}, {x, 1, 26}]
```

Out[144]= $x - \frac{x^2}{2} + \frac{x^3}{3}$

Out[145]= $-\frac{6}{(1+x)^4}$

Out[147]= {0.693147180559945, 0.833333333333333}

Out[148]= {3.29583686600433, 5546.66666666667}

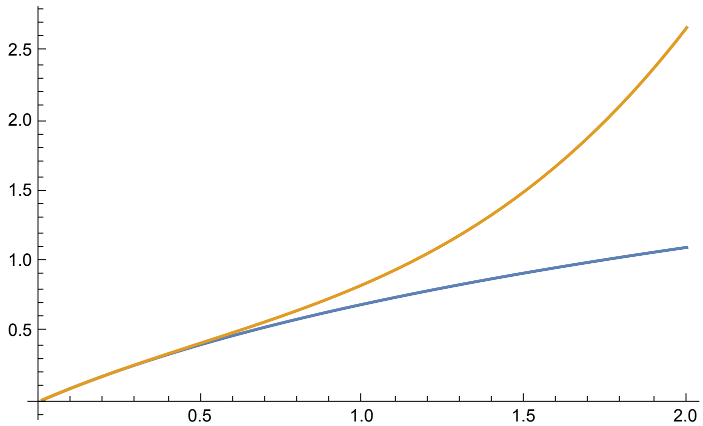
Out[149]= $\left\{\frac{1}{4}, 114\,244\right\}$

Out[150]= 0

Out[151]= 0.140186152773388

Out[152]= 5543.37082980066

Out[153]=



Out[154]=