#17 p. 142 Find the general approximation formula for the integral using two nodes, by doing the following:

(a) Write down the linear interpolating polynomial with nodes $x_0 + \theta_2 h$ & $x_0 + \theta_3 h$

(b) Integrate the polynomial over the interval $[x_0 + \theta_0 h, x_0 + \theta_1 h]$

(c) Simplify.

(a)

$$N_1(x) = y_2 + \frac{y_3 - y_2}{(x_0+\theta_3 h)-(x_0+\theta_2 h)} \cdot \left(x - (x_0+\theta_2 h)\right)$$

let $x = x_0 + \theta h$:

$$= y_2 + \frac{y_3 - y_2}{(\theta_3-\theta_2)h} \cdot \left(x_0+\theta h - (x_0+\theta_2 h)\right)$$

$$= y_2 + \frac{y_3 - y_2}{(\theta_3-\theta_2)h} \cdot (\theta-\theta_2)h$$

$$= y_2 + \frac{y_3 - y_2}{\theta_3-\theta_2} (\theta-\theta_2)$$

$$= N_1(x(\theta))$$

(b) $$I = \int_{x_0+\theta_0 h}^{x_0+\theta_1 h} N_1(x)\,dx$$

Let $x = x_0 + \theta h$

$$\frac{dx}{d\theta} = h \to dx = h\,d\theta$$

$$\therefore I = \int_{\theta_0}^{\theta_1} N_1(x(\theta))\,h\,d\theta$$

$$= h\int_{\theta_0}^{\theta_1}\left(y_2 + \frac{y_3-y_2}{\theta_3-\theta_2}(\theta-\theta_2)\right)d\theta$$

$$= h\left[y_2(\theta_1-\theta_0) + \left[\frac{1}{2}\frac{y_3-y_2}{\theta_3-\theta_2}(\theta-\theta_2)^2\right|_{\theta_0}^{\theta_1}\right]$$

$$= h\left[y_2(\theta_1-\theta_0) + \frac{1}{2}\frac{y_3-y_2}{\theta_3-\theta_2}\underbrace{\left[(\theta_1-\theta_2)^2-(\theta_0-\theta_2)^2\right]}\right]$$

(c) $$I = h\left[y_2(\theta_1-\theta_0) + \frac{y_3-y_2}{2(\theta_3-\theta_2)}\left[\theta_1^2-\theta_0^2 - 2\theta_2(\theta_1-\theta_0)\right]\right]$$

$$= \frac{h(\theta_1-\theta_0)}{2(\theta_3-\theta_2)}\left[2y_2(\theta_3-\theta_2) + (y_3-y_2)(\theta_1+\theta_0-2\theta_2)\right]$$

$$= \frac{h(\theta_1-\theta_0)}{2(\theta_3-\theta_2)}\left[(2\theta_3-\theta_1-\theta_0)y_2 - (2\theta_2-\theta_1-\theta_0)y_3\right]$$

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