Homework 4.1: Numerical Calculus

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pp. 140--, #4, 5, 6, 7, 8, 9(a-d), 10(a-d), 11, 12, 17

We did #4, 5, 6, and 7 together in class, so I'm not so worried about those. In addition, we did #17 together. So let's look at the rest:

#8

8. Formula 4.1.5 and the formula you got from question 4 should be different. However, they were derived over essentially the same stencil—two nodes with the point of evaluation centered between them. Only the labels on the stencils were different. In other words, they were derived from the same geometry, so, in some sense, must be the same. In question 4, x_0 plays the same role as $x_0 - h$ does in 4.1.5. Moreover, in question 4, the distance from the point of evaluation to either node is $\frac{h}{2}$ while in 4.1.5, that distance is h. Make the substitution x_0 for $x_0 - h$ in 4.1.5. Then make the substitution $\frac{h}{2}$ for the h in the denominator of 4.1.5. With these substitutions, formula 4.1.5 should match exactly the formula you got in question 4. In other words, different labelings in a stencil produce different labelings in the associated formula. Nothing more.

$$\begin{aligned} P_2'(x_0) &= \frac{1}{h(-1)(-2)} f(x_0 - h) + \frac{1 + (-1)}{h(1)(-1)} f(x_0) + \frac{1}{h(2)(1)} f(x_0 + h) \\ &= \frac{f(x_0 + h) - f(x_0 - h)}{2h}, \end{aligned} \tag{4.1.5}$$

In[430]:=
```
Clear[f, x0, h]
FourOneFive[f_, x0_, h_] := (f[x0 + h] - f[x0 - h]) / (2 h)
FourOneFive[f, x0, h]
```

Out[432]= $\frac{-f[-h+x0]+f[h+x0]}{2\,h}$

In[433]:=
```
FourOneFive[f, x0 + h / 2, h / 2]
```

Out[433]= $\frac{-f[x0]+f[h+x0]}{h}$

#9/10 (a-d)

9. Use formula 4.1.6 to approximate the integral.

(a) $\int_{-4}^{3} e^x \, dx$ [A]

(b) $\int_{-1}^{6} \sin x \, dx$

(c) $\int_{10}^{17} \frac{1}{x-5} \, dx$ [S]

(d) $\int_{-3}^{4} \left(x^5 - 4\right) dx$

$$\int_{x_0-h}^{x_0+6h} f(x)dx \approx \frac{h}{8640}[5257f(x_0+6h) - 5880f(x_0+5h) + 59829f(x_0+4h) \\ -81536f(x_0+3h) + 102459f(x_0+2h) - 50568f(x_0+h) + 30919f(x_0)]. \quad (4.1.6)$$

10. For each integral in question 9, (i) calculate the integral exactly, and (ii) calculate the absolute error in the approximation. [S] [A]

In[434]:=
```
FourOneSix[f_, x0_, h_] :=
  h / 8640 * (5257 f[x0 + 6 h] - 5880 f[x0 + 5 h] + 59 829 f[x0 + 4 h] - 81 536 f[x0 + 3 h] +
     102 459 f[x0 + 2 h] - 50 568 f[x0 + h] + 30 919 f[x0])
```

9/10a: Answer 20.32712878304436

In[435]:=
```
f[x_] := E^x
{a, b} = {-4, 3};
h = (b - a) / 7.0;
nineA = FourOneSix[f, a + h, h]
Integrate[f[x], {x, a, b}]
nineAexact = N[%]
Abs[nineA - nineAexact]
```

Out[438]= 20.3271287830444

Out[439]= $\frac{-1 + e^7}{e^4}$

Out[440]= 20.0672212842989

Out[441]= 0.259907498745431

9b:

In[442]:=
```
f[x_] := Sin[x]
{a, b} = {-1, 6};
h = (b - a) / 7.0;
nineB = FourOneSix[f, a + h, h]
Integrate[f[x], {x, a, b}]
nineBexact = N[%]
Abs[nineB - nineBexact]
```

Out[445]= -0.231633068587159

Out[446]= Cos[1] - Cos[6]

Out[447]= -0.419867980782226

Out[448]= 0.188234912195068

9c:

In[449]:=
```
f[x_] := 1 / (x - 5)
{a, b} = {10, 17};
h = (b - a) / 7.0;
nineC = FourOneSix[f, a + h, h]
Integrate[f[x], {x, a, b}]
nineCexact = N[%]
Abs[nineC - nineCexact]
```

Out[452]= 0.875396295127198

Out[453]= Log$\left[\frac{12}{5}\right]$

Out[454]= 0.8754687373539

Out[455]= 0.0000724422267017744

9d:

In[456]:=
```
f[x_] := (x^5 - 4)
{a, b} = {-4, 3};
h = (b - a) / 7.0;
nineD = FourOneSix[f, a + h, h]
Integrate[f[x], {x, a, b}]
nineDexact = N[%]
Abs[nineD - nineDexact]
```

Out[459]= -589.166666666667

Out[460]= $-\frac{3535}{6}$

Out[461]= -589.166666666667

Out[462]= $1.13686837721616 \times 10^{-13}$

#11

11. Let $f(x) = (x-1)^2 \sin x$. Use formula 4.1.4 to approximate $f'(0)$ using

   (a) $h = 1$
   (b) $h = \frac{1}{2}$ [A]
   (c) $h = \frac{1}{4}$
   (d) $h = \frac{1}{8}$

12. Calculate the absolute error in each approximation of question 11. Does the error get smaller as h gets smaller? [A]

$$P_2'\left(x_0 - \frac{1}{6}h\right) = \frac{-\frac{1}{6} - \frac{7}{6}}{h(-1)(-2)} f(x_0 - h) + \frac{\frac{5}{6} - \frac{7}{6}}{h(1)(-1)} f(x_0) + \frac{\frac{5}{6} - \frac{1}{6}}{h(2)(1)} f(x_0 + h) = \frac{-2f(x_0 - h) + f(x_0) + f(x_0 + h)}{3h}. \quad (4.1.4)$$

In[463]:=
```
FourOneFour[f_, x0_, h_] := (-2 f[x0 - h] + f[x0] + f[x0 + h]) / (3 h)
f[x_] := (x - 1)^2 Sin[x]
elevenExact = f'[0]
```

Out[465]= 1

11/12a:

```
In[466]:= h = 1.0;
elevenA = FourOneFour[f, h / 6, h]
Abs[elevenA - elevenExact]

Out[467]= 1.70545969231509

Out[468]= 0.705459692315094
```

11/12b:

11b: 1.19336533331362

12b: .19336533331362

```
In[469]:= h = 0.5;
elevenB = FourOneFour[f, h / 6, h]
Abs[elevenB - elevenExact]

Out[470]= 1.19336533331362

Out[471]= 0.19336533331362
```

11/12c:

```
In[472]:= h = 0.25;
elevenC = FourOneFour[f, h / 6, h]
Abs[elevenC - elevenExact]

Out[473]= 1.04866722403006

Out[474]= 0.0486672240300599
```

11/12d:

```
In[475]:= h = 0.125;
elevenD = FourOneFour[f, h / 6, h]
Abs[elevenD - elevenExact]

Out[476]= 1.0120901793354

Out[477]= 0.012090179335396
```

Just getting better and better and better with decreasing h!

In[478]:=
```
h = h / 2;
elevenD = FourOneFour[f, h / 6, h]
Abs[elevenD - elevenExact]
h = h / 2;
elevenD = FourOneFour[f, h / 6, h]
Abs[elevenD - elevenExact]
h = h / 2;
elevenD = FourOneFour[f, h / 6, h]
Abs[elevenD - elevenExact]
h = h / 2;
elevenD = FourOneFour[f, h / 6, h]
Abs[elevenD - elevenExact]
h = h / 2;
elevenD = FourOneFour[f, h / 6, h]
Abs[elevenD - elevenExact]
h = h / 2;
elevenD = FourOneFour[f, h / 6, h]
Abs[elevenD - elevenExact]
```

Out[479]= 1.00300565636992

Out[480]= 0.00300565636991501

Out[482]= 1.00074884977632

Out[483]= 0.000748849776321281

Out[485]= 1.00018686373337

Out[486]= 0.000186863733371512

Out[488]= 1.00004667058941

Out[489]= 0.0000466705894108799

Out[491]= 1.00001166186985

Out[492]= 0.0000116618698535742

Out[494]= 1.00000291473844

Out[495]= $2.91473843772039 \times 10^{-6}$