Deflating Polynomials with Real Coefficients by Complex Conjugate Pairs

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Abstract

Suppose you've found a root $z = a + bi$, complex. Then if the coefficients of the polynomial p(x) are real, you've found a second root: the complex conjugate, $\overline{z}$.

Thus, we might as well deflate by the quadratic

$$(x - z)(x - \overline{z}) = x^2 - 2Re[z]x + z\overline{z}$$

Let's represent the coefficients of this quadratic by the triple [1,$\alpha \equiv 2Re[z]$, $\beta \equiv z\overline{z}$].

Let's factor p(x) of degree n as

$$p(x) = r(x)(x^2 - 2Re[z]x + z\overline{z}) + f(x, z, \overline{z}).$$

where r is of degree $n - 2$, and where f is linear in x. In the derivation of Newton's method for polynomials found in *Tea Time Numerical Analysis*, the form of this equation was

$$p(x) = r(x)(x - t) + p(t)$$

where $p(t)$ is the desired end result (we need its value for Newton's next iteration step). Here we're extracting two roots simultaneously: and so we have a linear function which we can express elegantly using Lagrange polynomials:

$$f(x, z, \overline{z}) = p(z)\frac{x - \overline{z}}{z - \overline{z}} + p(\overline{z})\frac{x - z}{\overline{z} - z}$$

For this problem, however, because both z and $\overline{z}$ are roots of p, **this term should be about zero**.

So this is the algorithm: we equate coefficients,

$$[p_n, p_{n-1}, p_{n-2}, \ldots, p_1, p_0] = [r_{n-2}, r_{n-3}, r_{n-4}, \ldots, r_1, r_0][1, \alpha, \beta]$$

and so arrive at the following recursive scheme for the computation of the coefficients of polynomial r:

$$\begin{cases} r_{n-2} = & p_n \\ r_{n-3} = & p_{n-1} - \alpha r_{n-2} \\ r_{n-4} = & p_{n-2} - \alpha r_{n-3} - \beta r_{n-2} \\ \vdots & \vdots \\ r_1 = & p_3 - \alpha r_2 - \beta r_3 \\ r_0 = & p_2 - \alpha r_1 - \beta r_2 \\ 0 = & p_1 - \alpha r_0 - \beta r_1 \\ 0 = & p_0 - \beta r_0 \end{cases}$$

The last two equations serve as a check: if we have truly found two roots, then it should be the case that

$$p_1 - \alpha r_0 - \beta r_1 \approx 0$$

and

$$p_0 - \beta r_0 \approx 0.$$

Let's look at some examples....