**Directions**: You may use your (hand-written) notes, and a graphing calculator (or the equivalent). Each problem is equally weighted. Do your best, and **good luck!**

## Preliminaries

You'll be working with the function $f(x) = x(x-1)e^x$. Below are given the first several derivatives, along with some root information, and a graph of the function:

$$f(x) = e^x\,(x(x-1))$$

$$f'(x) = e^x\left(x^2 + x - 1\right)$$

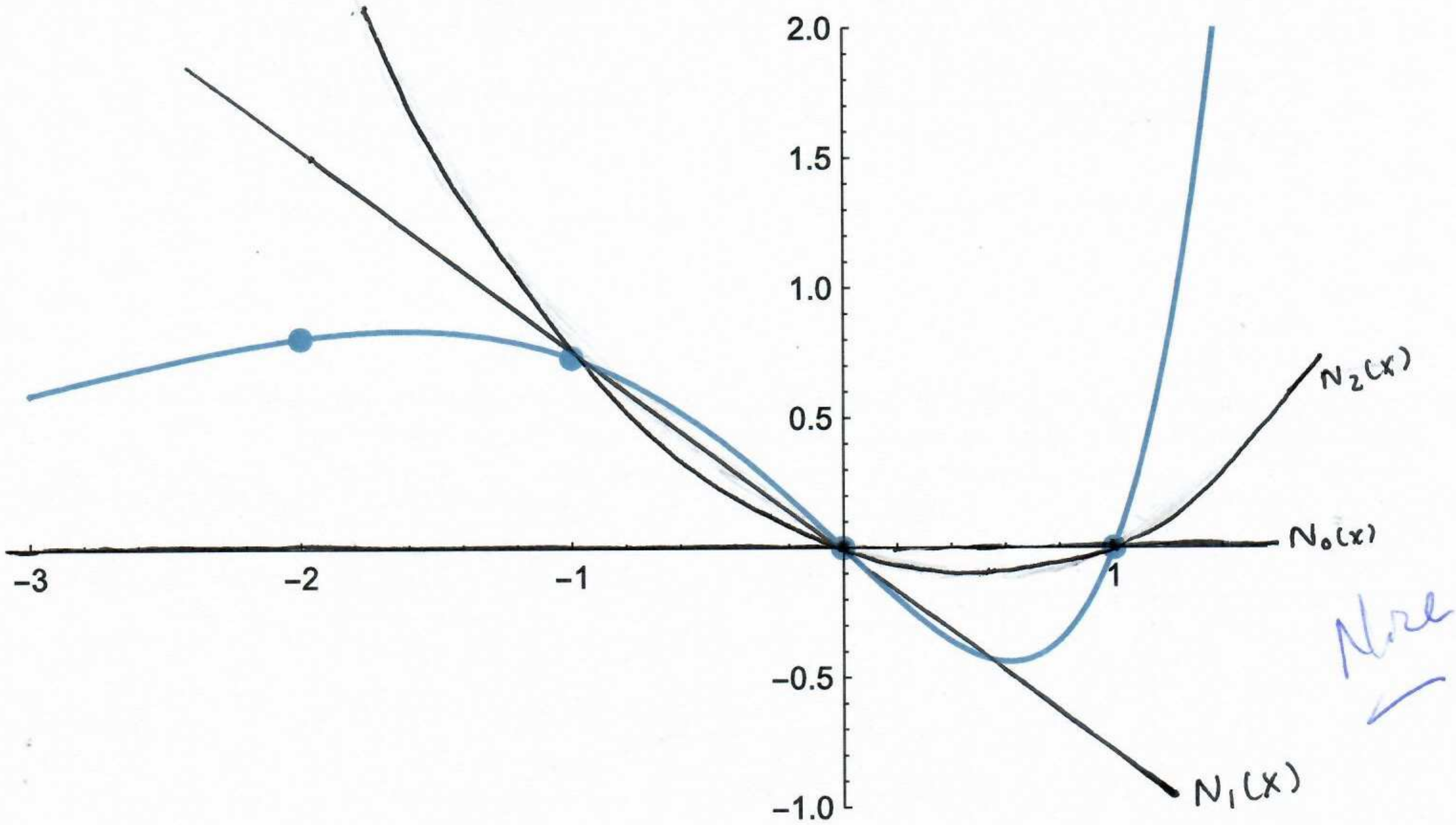
$$f''(x) = e^x\left(x^2 + 3x\right)$$

$$f^{(3)}(x) = e^x\left(x^2 + 5x + 3\right)$$

$$f^{(4)}(x) = e^x\left(x^2 + 7x + 8\right)$$

$$\begin{array}{l} \qquad\qquad\quad 4 \;\; -10 \; +3 \\ f^4(-2) = e^{-2}\left((-2)^2 + 5(-2) + 3\right) = -3/e^2 \\ f^4(1) = e(1+7+8) = 16e \end{array}$$

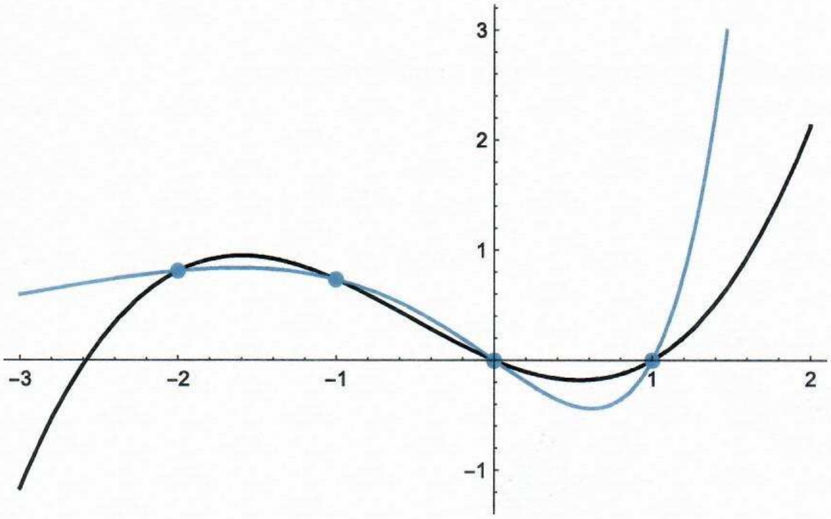
The roots of $f^{(5)}(x)$ are $r_1 \approx -6.8$ and $r_2 \approx -2.2$.



**Problem 1: Lagrange Polynomials** for $f(x) = x(x-1)e^x$

a. Using Lagrange form, write the equation of the cubic $L(x)$ to the function at the four points given at abscissa $x \in \{-2, -1, 0, 1\}$. You do not need to simplify coefficients. **Use exact arithmetic.**

A graph of $L(x)$ against $f(x)$ is below:

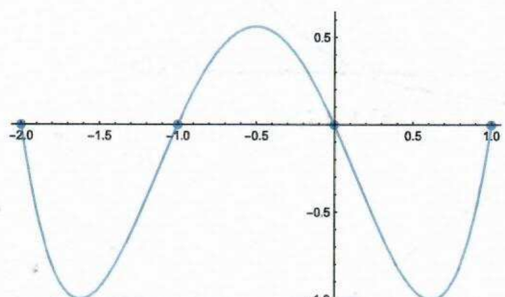



| $x_0$ | $f(x_0)$ |
|---|---|
| $-2$ | $-2(-2-1)e^{-2} = 6/e^2$ |
| $-1$ | $-1(-1-1)e^{-1} = 2/e$ |
| $0$ | $0(0-1)e^0 = 0$ |
| $1$ | $1(1-1)e^1 = 0$ |

$(-2, 6/e^2)$
$(-1, 2/e)$
$(0,0)$
$(1,0)$

$$\boxed{L(x) = \frac{(x+1)(x)(x-1)(6/e^2)}{(-2+1)(-2)(-2-1)} + \frac{(x+2)(x)(x-1)(2/e)}{(-1+2)(-1)(-1-1)}}$$

b. Use an error formula to bound the error of the interpolating polynomial as tightly as you can over the entire inverval [-2,1]. If it helps, you may use this graph of $(x-x_0)(x-x_1)(x-x_2)(x-x_3)$:




$|max| = 1$

$$f(x) - L(x) = \frac{e^{(\xi(x))}(\xi(x)^2 + 7(\xi(x)) + 8)}{4!}(x+2)(x+1)(x)(x-1)$$

$$f(x) - L(x) \le \max_{x\in[-2,1]} \left| \frac{e^x(x^2+7x+8)}{24} \right| \cdot \underbrace{\max_{x\in[2,1]} \left| (x+2)(x+1)(x)(x-1) \right|}_{\text{shown in graph}}$$

$\le (16e)(1)$

$\boxed{f(x) - L(x) \le 16e}$

$f^4(-2) = -3/e^2 \approx -.4$
$f^4(1) = 16e \approx 43.5$
And roots of $f^5(x)$ are @ $x = -6.8$ & $x = -2.2$ which aren't in $[2,1]$ which means no other max or min in $[2,1]$ ∴ $\max_{x\in[2,1]} = 16e$

good!

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c. Why do we not use Lagrange's form to evaluate polynomials?

Not the best way to numerically evaluate, not as numerically stable. Instead we use Nevilles Method to evaluate the polynomials for specific values.

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$-1(-1-1)e^{-1}$ $\quad 2/e$ $\qquad \frac{-2/e}{2}$

$-2(-2-1)e^{-2}$ $\quad 6/e^2$

**Problem 2: Newton Polynomials** for $f(x) = x(x-1)e^x$

Suppose that we want to use a succession of polynomials of increasing degree to approximate the function at $x = -0.5$. Use the Newton form of the interpolating polynomial and the same four points as above to create interpolators of each degree up to cubic.

a. For your divided difference table (you should be creating a divided difference table, right?), let's choose the points in this order. (What do you think of the choice?) **Use exact arithmetic!**

| $x_i$ | $f[x_i]$ | | | |
|---|---|---|---|---|
| $x_0 = 0$ | $0$ | $\frac{2/e}{-1-0} = -\frac{2}{e}$ | $\frac{-1/e-(-2/e)}{1-0} = \frac{1}{e}$ | $\frac{\frac{2-e}{e^2}-\frac{1}{e}}{(-2-0)} = \frac{-1+e}{e^2}$ |
| $x_1 = -1$ | $\frac{2}{e}$ | $\frac{0-2/e}{1-(-1)} = -\frac{1}{e}$ | $\frac{-2/e^2-(-1/e)}{-2-(-1)} = \frac{2-e}{e^2}$ | ✓ |
| $x_2 = 1$ | $0$ | $\frac{6/e^2-0}{-2-1} = -\frac{2}{e^2}$ | | |
| $x_3 = -2$ | $\frac{6}{e^2}$ | | | |

$\frac{6/e^2}{-3} = -\frac{2}{e^2}$ $\quad \frac{-1}{e}+\frac{2}{e}=\frac{1}{e}$ $\quad \frac{-2}{e^2}+\frac{e}{e^2}$ $\quad \frac{2-e}{e^2}-\frac{e}{e^2}$ $\quad \frac{2-2e}{-2(e^2)}$ $\quad \frac{-1+e}{e^2}$

b. Write the equation of your Newton polynomial $N_3(x)$. Can you also write it in the most efficient form for computation?

$N_3(x) = 0 - \frac{2}{e}(x-0) + \frac{1}{e}(x-0)(x+1) + \frac{-1+e}{e^2}(x-0)(x+1)(x-1)$

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c. Sketch the constant, linear, and quadratic functions you used into the graph on the first page. Nice!

$\frac{(-\frac{1}{2}x-\frac{1}{2}+1)}{e}$ $\quad \frac{4}{4e}-\frac{1}{4e}$ $\quad \frac{-\frac{3}{4}}{-e}$ $\quad \frac{3}{4e}$

d. Write their equations in this table, and compute their estimates for $f(-0.5)$ and the absolute error ($f(-0.5) \approx 0.454898$).

| degree | equation | estimate | absolute error |
|---|---|---|---|
| constant | $N_0(x) = 0$ | $0$ | .454898 |
| linear | $N_1(x) = \frac{-2x}{e}$ | $\frac{1}{e} \approx .3678794$ | .087019 |
| quadratic | $N_2(x) = \frac{-2x}{e} + \frac{x(x+1)}{e}$ ✓ | $\frac{3}{4e} \approx .275909$ | .178988 |
| cubic | given above | $\frac{7e-1}{8e^2} \approx .304977$ | .149921 |

$\frac{3}{4e} + \frac{-1+e}{e^2}(-\frac{1}{2})(\frac{1}{2})(-\frac{3}{2})$ $\quad \frac{3(2e)}{4e(2e)} + \frac{-1+e}{8e^2}$ $\quad \frac{6e}{8e^2}+\frac{-1+e}{8e^2}$ $\quad \frac{7e-1}{8e^2}$

e. Can you explain the pattern (or lack thereofo) in the absolute errors?

$f(-.5) \approx .454898$
$f(0) = 0$
$f(-1) = 2/e \approx .735758$
$f(1) = 0$
$f(-2) = .812011$

good

The equation with the least error was the linear because the function values at the x-values used were the two closest the value $f(-.5)$. When we added points $(1,0)$ and $(-2, 6/e^2)$, our error for the estimate using the quadratic and cubic increased, because the function values are actually farther from $f(-.5)$ than the linear estimate, so adding those points brought our estimate

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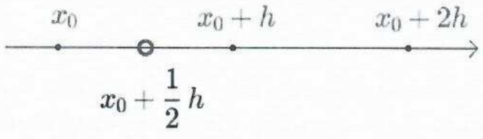
**Problem 3: Numerical Calculus**: Derive an approximation formula for the first derivative over the stencil




following these steps:

a. Calculate $N_2(x)$, the Newton form of the interpolating polynomial passing through the points $(x_0, f(x_0))$, $(x_1, f(x_1))$, and $(x_2, f(x_2))$.

$x_0 \quad f(x_0) \quad \frac{f(x_1)-f(x_0)}{x_1-x_0}$

$x_1 \quad f(x_1) \quad \frac{f(x_2)-f(x_1)}{x_2-x_1}$

$x_2 \quad f(x_2)$

$$\frac{\frac{f(x_1)-f(x_0)}{x_1-x_0} - \frac{f(x_2)-f(x_1)}{x_2-x_1}}{x_2-x_0} \leftarrow \text{Big Bad Func (BBF)}$$

$$N_2(x) = f(x_0) + \frac{f(x_1)-f(x_0)}{x_1-x_0}(x-x_0) + BBF\,(x-x_0)(x-x_1)$$

good!

b. Calculate the derivative $N_2'(x)$.

$$N_2'(x) = \frac{f(x_1)-f(x_0)}{x_1-x_0} + (2x - x_0 - x_1)\,BBF$$

c. Substitute $x_0 + \frac{1}{2}h$ for $x$, $x_0 + h$ for $x_1$, and $x_0 + 2h$ for $x_2$ in your formula from (b) and simplify.

$$\frac{f(x_0+h) - f(x_0)}{h} + \cancel{\overbrace{(2x_0 + h - x_0 - x_0 - h)}^{0}\,BBF}$$

d. Estimate $f'(-0.5)$ using this scheme, for $f(x) = x(x-1)e^x$.

$x_0 + \frac{1}{2}h = -0.5$

$h = 1$

$x_0 = -1$

$$\frac{f(0) - f(-1)}{1} = 2e^{-1} = 0.735758$$

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**roblem 4: Undetermined Coefficients** In Problem 3, we found a formula for the first derivative based on three points. In other words, we found three coefficients for the three points that give a good general formula for the derivative of $f(x_0 + \frac{1}{2}h)$. Let's find it in a different way: by making the formula exact for polynomials of degree two or less.

We seek coefficients $a$, $b$, and $c$ such that

$$f'(x_0 + \frac{1}{2}h) \approx af(x_0) + bf(x_0 + h) + cf(x_0 + 2h)$$

$x - x_0$

$x \quad x^2 \quad -x_0 x$

$-x_0 \quad -x_0 x$

is exact for constant, linear, and quadratic polynomials. Use the usual base polynomials:

$$p_0(x) = 1 \qquad p_1(x) = x - x_0 \qquad p_2(x) = (x - x_0)^2$$

a. First equation: make it **exact** for constant polynomials (when $f(x) = p_0(x)$):

$0 = a + b + c$

$P_0'(x) = 0 \qquad P_0(x) = 1$

b. Second equation: make it **exact** for linear polynomials (when $f(x) = p_1(x)$):

$P_1'(x) = 1 \qquad P_1(x) = x - x_0$

$1 = a(0) + b(h) + c(2h)$

c. Third equation: make it **exact** for quadratic polynomials (when $f(x) = p_2(x)$):

$P_2'(x) = 2x - 2x_0 \qquad P_2(x) = (x - x_0)^2$

$2x_0 + h - 2x_0 = h$

$h = a(0)^2 + b(h)^2 + c \cdot 4h^2$

Divided all by $h$

d. You have three equations for the three unknowns $a$, $b$, and $c$: use back-substitution to find them.

$0 = a + b + c$

$1 = bh + 2ch$

$1 = bh + 4ch$

$b = \frac{1}{h}$

$c = 0$

$a = -\frac{1}{h}$

e. What's your formula? (Hopefully you got the same thing! If not, which do you trust more?)

$f'(x_0 + \frac{1}{2}h) \approx -\frac{1}{h}f(x_0) + \frac{1}{h}f(x_0 + h)$

$\rightarrow \dfrac{f(x_0 + h) - f(x_0)}{h}$

the same!

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