Tricolorability of Knots

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ABSTRACT: Knot theory is an exciting area of study, with many applications in the sciences. After discussing the history of the subject and covering basic definitions, we'll discuss the property of tricolorability and prove its use in answering a fundamental question in knot theory: whether a given knot is equivalent to the unknot.

1. A BRIEF HISTORY OF KNOT THEORY

While knot theory is now considered a branch of topology, it originated not in mathematics but in chemistry. During the later 1800s, when Lord Kelvin and his physicist cronies were speculating on the Nature of the Universe, they hypothesized that perhaps atoms corresponded to knots in the ether, the mysterious substance thought to fill all space. In the pursuit of creating a table of the elements, they cataloged different kinds of knots by trial and error – quite a daunting task. (See, for example, the partial table in the attached Appendix.)

Unfortunately for their theory, the famous Michelson-Morley of the late nineteenth century dashed scientists' hopes for finding knots in the ether, by showing there *was* no ether. The chemists lost interest and abandoned the field in favor of concentrating on newer, more promising atomic models. Luckily, mathematicians' curiosity was by now piqued, and knot theory continued to be studied – just in a different department.

Ironically, recently chemists have renewed their involvement in the subject after knotting was found to be an important property of DNA and synthetic molecules. Mathematicians have been gracious about allowing chemists to reap from the results of their interim century of study, and knot theory is now an active and intriguing field of research, as much for its many applications to physics and chemistry as for its for its beautiful and fascinating theory.

2. KNOT FUNDAMENTALS

So what exactly is a knot? There are a number of possible definitions that could formalize what we mean, and we'll work with an easy one:

> **DEFINITION 1:** *A* ***knot*** *is a simple, closed, non-self-intersecting curve in* R^3.

For physical intuition, you can think of a knot as constructed from a string glued together at the ends, perhaps tangled in the middle. Thus your shoelaces' knot wouldn't count, but it would if you connected the loose ends together. We won't concern ourselves with the thickness of our string – its cross-section will be just a point.

One way of graphically representing knots, called a **knot diagram** or **projection**, shows the curve as a black line on the page. Formally, the projection is the image of a function from the R^3 in which the knot lives to the plane of the page, taking the triple (x, y, z) to the pair (x, y).

When the knot passes under itself, we leave a gap in the picture.

Some more rules about knots: we won't distinguish between a knot and any deformations of it through space so long as we never cut the string or pass the string through itself. The picture we draw to represent it will certainly look different, but the knot itself is unchanged. A knot deformation is a planar isotopy: we can bend and stretch and twist the projection plane as if it were (very flexible) rubber, with the knot drawn on the rubber (see Figure 1). However, we are not allowed to shrink part of the knot down to a point to rid ourselves of it.

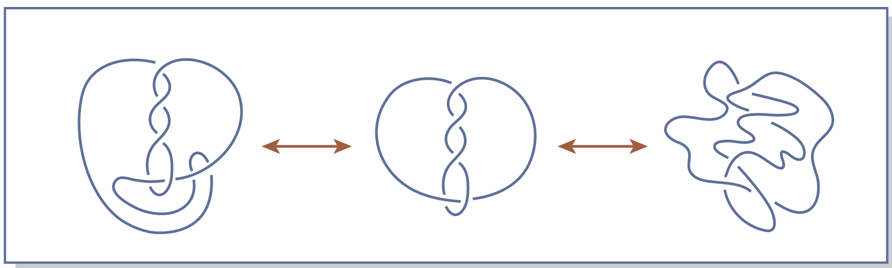

Figure by MIT OCW.

Figure 1: A knot is unchanged by deformities.

Two further definitions will be helpful:

DEFINTITION 2: *A* ***crossing*** *in a knot diagram is a place where the knot curve crosses – going over or under – itself.*

DEFINITION 3: *An* ***arc*** *in a knot diagram is a piece of the curve going between two undercrossings. Overcrossings are allowed along the way.*

Figure 2 below shows the three simplest knots – the kinds that can be drawn with the least number of crossings. The leftmost is the **unknot**, or **trivial knot**, which can be drawn with no crossings. The middle is a **trefoil knot**, the simplest nontrivial knot, with a minimum of three crossings. The rightmost picture is of a **figure-eight knot**, so called because of the 8 shape in the middle, which can be drawn with a minimum of four crossings.

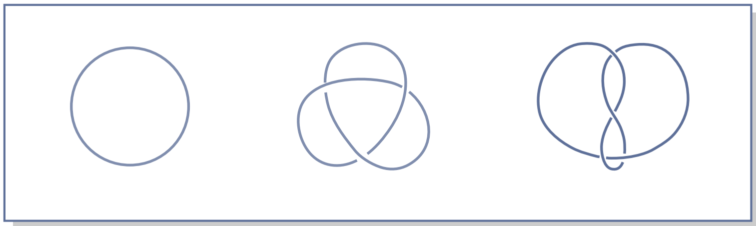

Figure by MIT OCW.

Figure 2: Three fewest-crossing knots: the unknot (0 crossings), a trefoil knot (3 crossings), and the figure-eight knot (4 crossings).

Of course, there are arbitrarily many ways of drawing a knot's diagram, with as many more crossings as you'd like, given our allowance of deformations. In Figure 3 you can see three

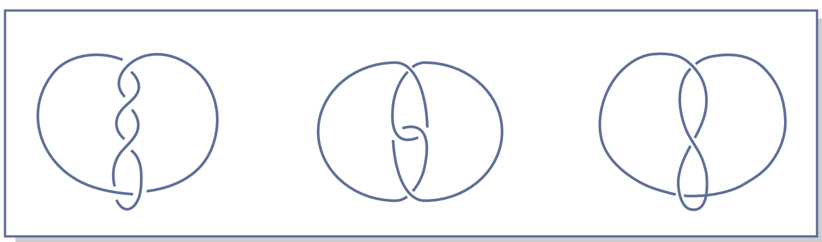

Figure by MIT OCW.

Figure 3: Three diagrams of a figure-eight knot.

equivalent diagrams of a figure-eight knot:

Perhaps it's not too difficult to see how the three diagrams are of the same knot in the picture above, but sometimes it can be a much harder task. For example, recognizing the knot in Figure 4 as equivalent to the unknot can look daunting:

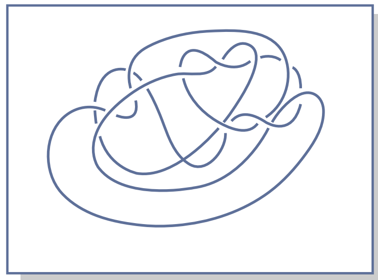

Figure by MIT OCW.

Figure 4: A (very) messy diagram of the unknot.

Knot equivalence is in general a difficult thing to prove, as we'll see shortly. But it will be critical in working towards the solution of one of the central problems of knot theory:

THE BIG QUESTION: *Is a given knot the unknot?*

This question has actually been answered – sort of. In 1961, Haken (the same fellow who 13 years later would co-publish a revolutionary computational proof of the Four-Color Theorem [Appel & Haken 1997]) devised a 130-page algorithm [Haken 1961] for determining whether a given knot is the unknot. However, *implementing* this algorithm on a computer, even for relatively simple cases, continues to elude researchers, and so our Big Question remains an open problem.

Until now, we have not discussed any ways to tell two different diagrams are of equivalent knots. Let's turn now to a contribution of Kurt Reidemeister.

3. REIDEMEISTER MOVES

We start by defining three simple allowed ways to deform knot diagrams by changing the number of crossings, called **Reidemeister moves** (see Figure 5 below):

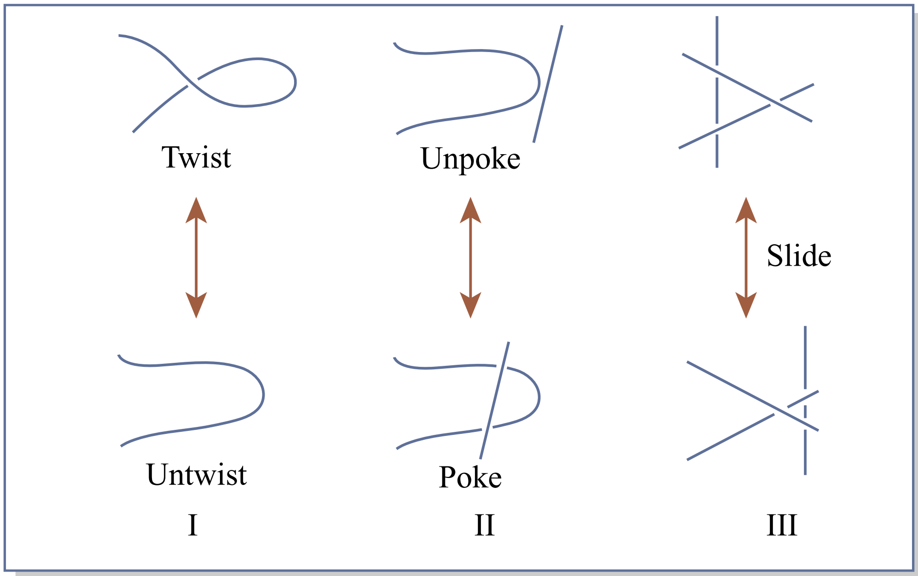

Figure by MIT OCW.

Figure 5: The three Reidemeister moves, ways to change knot diagrams. Note that these are the only allowed moves that change the relation between crossings.

These three moves are not very complicated, but can generate *any* valid deformation that keeps the knot the same:

> **THEOREM 1:** *Two knots are equivalent if their diagrams are related by a sequence of Reidemeister moves.*

Of course, it may not be obvious *which* Reidemeister moves you should use and where, but the point is that it's possible. For example, Figure 6 demonstrates that a figure-eight knot is equivalent to its mirror image, by showing a short sequence of Reidemeister moves:

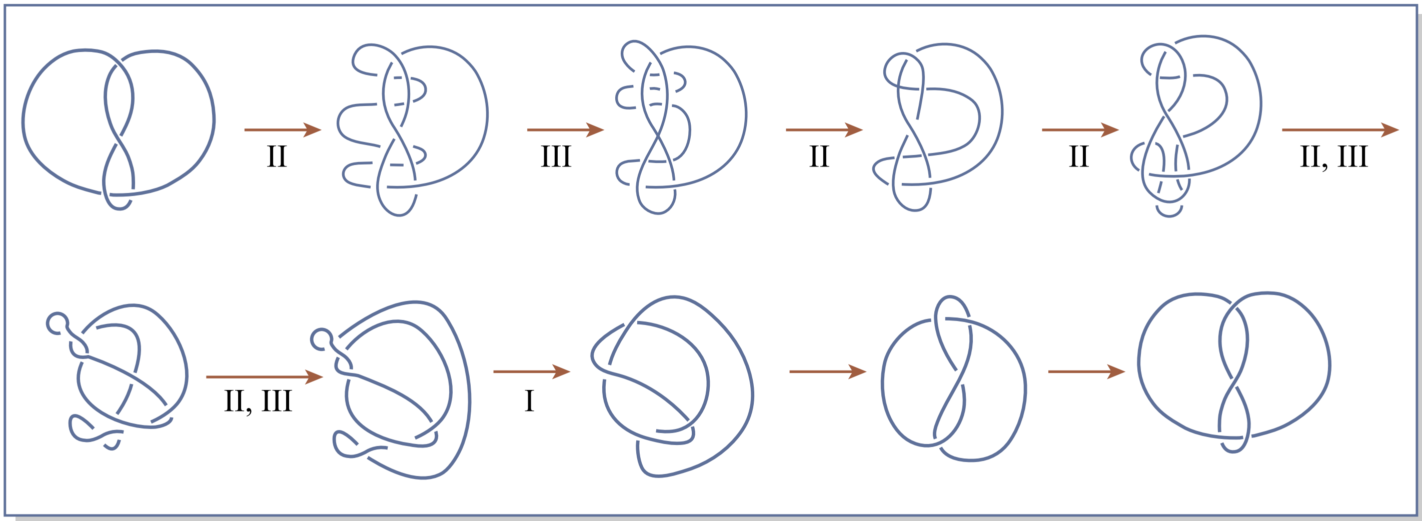

Figure by MIT OCW.

Figure 6: The figure-eight knot is equivalent to its mirror image, as shown through a sequence of Reidemeister moves.

(Interestingly, a trefoil knot is *not* equivalent to its mirror image – no amount of pushing the loops around will ever make transform one into the other.)

4. TRICOLORABILITY

One valuable method of distinguishing the unknot from other knots is by using a diagram's tricolorable property:

DEFINITION 4: *A knot diagram is* ***tricolorable*** *(sometimes called simply* ***colorable****) if each arc can be colored with one of three colors such that:*

(1) at least two colors are used in the diagram; and

(2) at every crossing, either:

(i) all three colors are used; or

(ii) only one color is used.

For example, the trefoil knots shown in Figure 7 have been tricolored with white, black, and gray. The left knot has three colors at every crossing, while the right has some crossings where only one color is used:

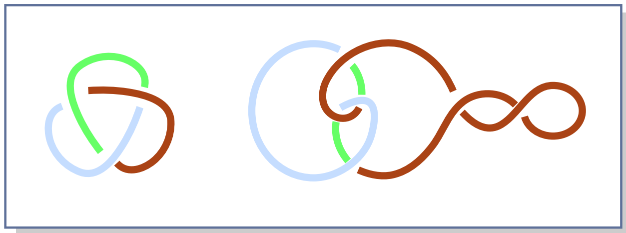

Figure by MIT OCW.

Figure 7: Tricolored trefoil diagrams.

If we use the integers 0, 1, and 2 as color labels, and at each crossing mark the overcrossing as x and the other two arcs as y and z, then we can restate requirement (2) as:

$$2x - y - z = 0 \pmod 3$$

Very nice, you may be wondering, but so what? The power lies in the following seemingly-innocent theorem, a proof of which we'll sketch shortly:

> **THEOREM 2:** *Tricolorability is invariant under Reidemeister moves.*

In other words, if a given diagram of a knot is tricolorable, then every diagram of the same knot is tricolorable. The following definition thus follows readily:

> **DEFINITION 5:** *A knot is* ***tricolorable*** *if its diagrams are tricolorable.*

Note that this is only a statement about *whether* a knot can be tricolored; *how* a tricolorable knot is tricolored can vary depending on which of its diagrams you're looking at (and indeed, some diagrams can be tricolored in more than one way).

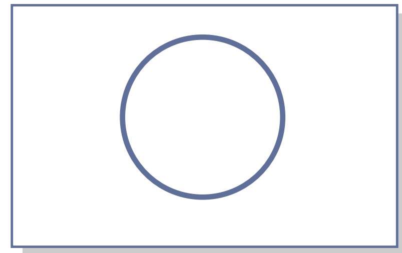

Figure by MIT OCW.

Figure 8: The unknot can't be tricolored.

Now we're finally read to see why tricoloring helps us distinguish knots. First, consider the unknot. Its standard diagram, a simple loop with no crossings (Figure 8), cannot be tricolored since only once color can be assigned to its one arc, violating requirement (1). Therefore, *every tricolorable knot is nontrivial*. If we can tricolor a given diagram, we can confidently say that it represents a knot other than the unknot. And since we can certainly tricolor some diagrams, then we have just *proven the existence of nontrivial knots!*

In order to properly enjoy this fine consequence, we'll now outline the theorem's justification.

PROOF OF THEOREM 2: Following the method outlined in Livingston [p. 33-35] and Adams [p. 24-27], we will show that a tricolorable diagram remains tricolorable under a Reidemeister I and II moves. (Checking the remaining Reidemeister III moves can be done by similar reasoning, and left as an exercise for the motivated reader.)

(I) Suppose we perform a Reidemeister I move (*untwist* → *twist*) on a section of an already-tricolored diagram. We can preserve coloration by simply making the arcs involved all the same color (see Figure 9 for affected region). Similarly, the inverse (*twist* → *untwist* to remove a crossing) also preserves tricolorability; because there are only two arcs involved in a twist, they must be the same color, so the untwisted arc just stays that color.

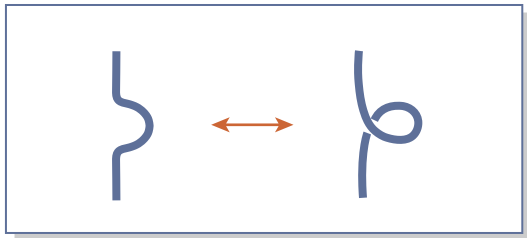

Figure by MIT OCW.

Figure 9: A Reidemeister I move preserves tricolorability.

(II) Now let's check the Reidemeister II moves. Perform a *poke* → *unpoke* and check the two possible cases: Either only one color is used for all affected arcs, in which case we keep the resulting two arcs the same color; or two (and thus all three) colors are used. In this situation, we simply remove one of the colors, as shown in Figure 10.

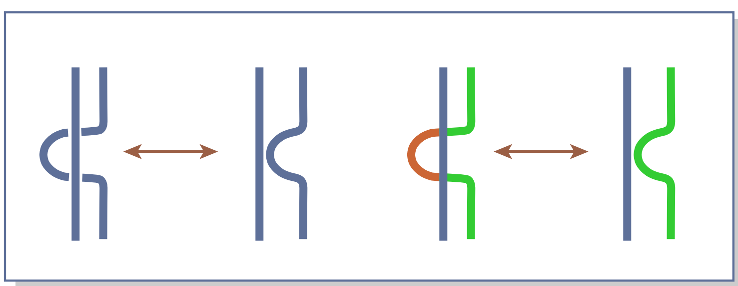

Figure by MIT OCW.

Figure 10: A Reidemeister II move preserves tricolorability for both possible original colorings.

The inverse, *unpoke* → *poke* is also straight-forward: if both original strands are the same color, keep them that way; if they're different, color the new arc with the third unused color.

Wonderful as this all is, let us pause to emphasize what Theorem 2 does and does not tell us. As we mentioned earlier, tricolorability can distinguish conclusively between the trefoil knot and the unknot, and indeed between any tricolorable and any non-tricolorable knots. However, this proves merely the existence of at least *two* knots: we are not able, for example, to demonstrate that the figure-eight knot is different from the unknot using this method. But it's a start.

Happily, knot theory is a large and expanding area of study, with many similarly easy-to-state-and-hard-to-prove problems that are still open for exploration. Enjoy!

APPENDIX

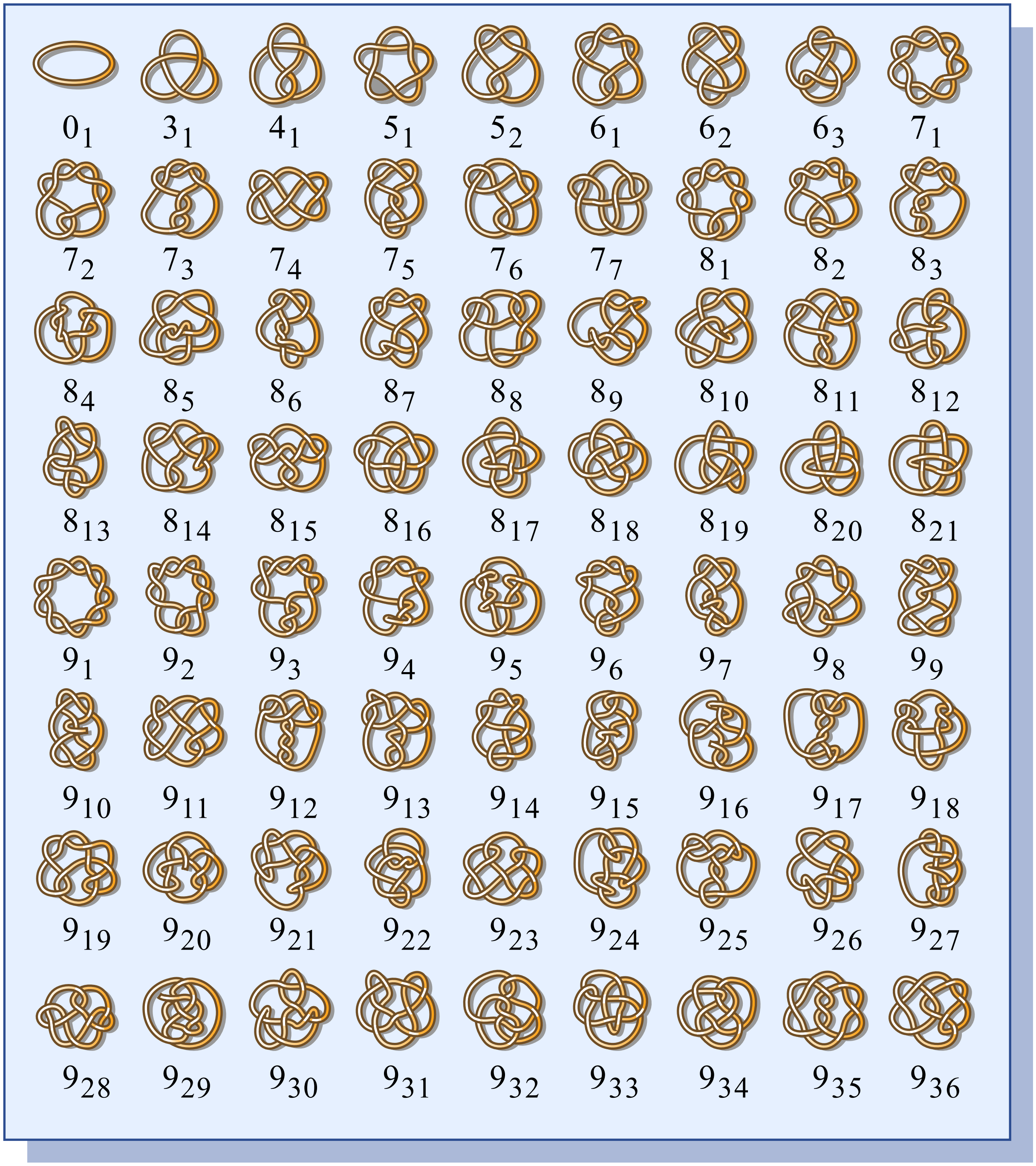

Figure by MIT OCW.

Figure 11: Knot classification by minimal crossing number. Note that there are 13 more possible knots than shown here with minimal crossing number nine.

JUST FOR FUN: A Wonderful Knot Joke [Adams p. 276]

“Marty Scharlemann tells the story of a calculus student who came in for help, and after Marty had worked some problems, the student said, ‘So what kind of math do you like?’

Marty said, ‘Knot theory.’

The student said, ‘Yeah, me either.’“

REFERENCES & RECOMMENDED FURTHER READINGS

Adams, Colin C. The Knot Book: An Elementary Introduction to the Mathematical Theory of Knots. Providence, RI: American Mathematical Society, 2001.

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