$$F(x) = \frac{2a(x) - 5b(x)}{c(x) \cdot d(x)},$$

then F is a quotient, in which the numerator is a sum of constant multiples and the denominator is a product. This, the derivative of F can be found by applying the quotient rule and then using the sum and constant multiple rules to differentiate the numerator and the product rule to differentiate the denominator.

2.3.5 Exercises

1. Derivative of a basic product.

Activate

Find the derivative of the function $f(x)$, below. It may be to your advantage to simplify first.

$f(x) = x \cdot 13^x$ $\Rightarrow$ product

$f'(x) = (x \cdot 13^x) = (x)' \cdot 13^x + x \cdot (13^x)'$

$= 1 \cdot 13^x + x \cdot \ln(13) 13^x$

$\ln(13)$

$f'(x) =$ $13^x(1 + \ln(13)x)$

2. Derivative of a product.

Activate

Find the derivative of the function $f(x)$, below. It may be to your advantage to simplify first.

$f(x) = (x^9 - \sqrt[3]{x})2^x = (x^9 - x^{1/3}) \cdot 2^x$

$f'(x) = (x^9 - x^{1/3})' \cdot 2^x + (x^9 - x^{1/3}) \cdot (2^x)'$

$= (9x^8 - \frac{1}{3}x^{-2/3}) \cdot 2^x + (x^9 - x^{1/3}) \cdot \ln(2) 2^x$

$f'(x) =$ $2^x(9x^8 - \frac{1}{3}x^{-2/3} + x^9 - x^{1/3})$

3. Derivative of a quotient of linear functions.

Activate

Find the derivative of the function z, below. It may be to your advantage to simplify first.

$$z = \frac{2t + 7}{8t + 7}$$

$z' = \frac{(2t+7)'(8t+7) - (2t+7)(8t+7)'}{(8t+7)^2}$

$= \frac{2(8t+7) - (2t+7)8}{(8t+7)^2}$

$\frac{dz}{dt} =$ $\frac{77}{(8t+7)^2}$

4. Derivative of a rational function.

Activate

Find the derivative of the function $h(r)$, below. It may be to your advantage to simplify first.

$$h(r) = \frac{r^3}{9r + 13}$$

$h'(r) = \frac{(r^3)'(9r+13) - r^3(9r+13)}{(9r+13)^2}$

$= \frac{3r^2(9r+13) - r^3 \cdot 9}{(9r+13)^2}$

$h'(r) =$ $\frac{18r^3 + 39r^2}{(9r+13)^2} = \frac{3r^2(6r+13)}{(9r+13)^2}$

5. Derivative of a product of trigonometric functions.

Activate

Find the derivative of $s(q) = 6 \cos q \sin q$.

$s'(q) = 6[\cos'(q)\sin(q) + \cos(q)\sin'(q)]$

$= 6[(-\sin(q))\sin(q) + \cos(q)(\cos(q))]$

$s'(q) =$ $6(-\sin^2 q + \cos^2 q)$

6. Derivative of a product of power and trigonmetric functions.

Activate

Find the derivative of $f(x) = x^5 \cos x$

$f'(x) = (x^5)'\cos(x) + x^5 \cdot (\cos(x))'$

$= 5x^4\cos(x) + x^5 \cdot (-\sin(x))$

$f'(x) =$ $x^4(5\cos(x) - x\sin(x))$

7. Derivative of a sum that involves a product.

Activate

Find the derivative of $h(t) = t \sin t + \tan t$

$h'(t) = (t\sin(t))' + (\tan(t))'$

$= t'\sin(t) + t(\sin(t))' + \frac{1}{(\cos(t))^2}$

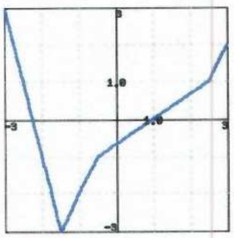
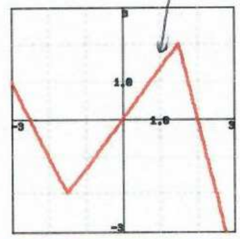
$h'(t) =$ $\sin(t) + t\cos(t) + \frac{1}{(\cos(t))^2}$

8. Product and quotient rules with graphs.

Activate

Let $h(x) = f(x) \cdot g(x)$, and $k(x) = f(x)/g(x)$. Use the figures below to find the ***exact*** values of the indicated derivatives.

$m = \frac{2}{3/2} = \frac{4}{3}$

	
$f(x)$	$g(x)$

$k'(-2) = \frac{f'(-2)g(-2) - f(-2)g'(-2)}{(g(-2))^2} = \frac{-4\cdot(-1) - (-1)(-2)}{(-1)^2} = 2$

A. $h'(1) =$ $4/3$

B. $k'(-2) =$ 2

$h'(x) = f'(x)g(x) + f(x)g'(x)$

$h'(1) = f'(1)g(1) + f(1)g'(1)$

$= 1 \cdot \frac{4}{3} + 0 \cdot \frac{4}{3} = \frac{4}{3}$

*(Enter **dne** for any answer where the derivative does not exist.)*

9. Product and quotient rules with given function values.

Activate

Let $F(4) = 4, F'(4) = 5, H(4) = 4, H'(4) = 5$.

$G'(z) = F'(z)H(z) + F(z)H'(z)$

$G'(4) = F'(4)H(4) + F(4)H'(4)$

$= 5 \cdot 4 + 4 \cdot 5 = 40$

A. If $G(z) = F(z) \cdot H(z)$, then $G'(4) =$ $\boxed{40}$

B. If $G(w) = F(w)/H(w)$, then $G'(4) =$ $\frac{F'(4)H(4) - F(4)H'(4)}{(H(4))^2} = \frac{5\cdot 4 - 4\cdot 5}{(4)^2} = \boxed{0}$

10. Let f and g be differentiable functions for which the following information is known: $f(2) = 5$, $g(2) = -3$, $f'(2) = -1/2$, $g'(2) = 2$.

a. Let h be the new function defined by the rule $h(x) = g(x) \cdot f(x)$. Determine $h(2)$ and $h'(2)$.

$h'(x) = g'(x)f(x) + g(x)f'(x)$

$h'(2) = g'(2)f(2) + g(2)f'(2)$

$= 2 \cdot 5 + (-3)(-\frac{1}{2})$

$= \frac{23}{2}$

$h(2) = g(2)f(2) = (-3)\cdot 5 = -15$

b. Find an equation for the tangent line to $y = h(x)$ at the point $(2, h(2))$ (where h is the function defined in (a)).

$L(x) = -15 + \frac{23}{2}(x-2)$

c. Let r be the function defined by the rule $r(x) = \frac{g(x)}{f(x)}$. Is r increasing, decreasing, or neither at $a = 2$? Why? $\boxed{r'(x) > 0}$

$r'(x) = \frac{g'(x)f(x) - g(x)f'(x)}{g(x)^2}$

$r'(2) = \frac{2 \cdot 5 - (-3)(-\frac{1}{2})}{5^2}$

$= \frac{17/2}{25} = \frac{17}{50}$

d. Estimate the value of $r(2.06)$ (where r is the function defined in (c)) by using the local linearization of r at the point $(2, r(2))$.

$L(2.06) = -15 + \frac{23}{2}(2.06 - 2) = -15 + \frac{23}{2}(0.06)$

11. Consider the functions $r(t) = t^t$ and $s(t) = \arccos(t)$, for which you are given the facts that $r'(t) = t^t(\ln(t) + 1)$ and $s'(t) = -\frac{1}{\sqrt{1-t^2}}$. Do not be concerned with where these derivative formulas come from. We restrict our interest in both functions to the domain $0 < t < 1$.

a. Let $w(t) = t^t \arccos(t)$. Determine $w'(t)$.

b. Find an equation for the tangent line to $y = w(t)$ at the point $(\frac{1}{2}, w(\frac{1}{2}))$.

c. Let $v(t) = \frac{t^t}{\arccos(t)}$. Is v increasing or decreasing at the instant $t = \frac{1}{2}$? Why?

12. Let functions p and q be the piecewise linear functions given by their respective graphs in Figure 2.3.6. Use the graphs to answer the following questions.

$r'(x) = p'(x)q(x) + p(x)q'(x)$ $\quad$ $r'(-2) = p'(-2)q(-2) + p(-2)q'(-2)$

$= (-2)(-1) + 1\cdot 6$

$\approx 2 + 6 = \boxed{8}$

a. Let $r(x) = p(x) \cdot q(x)$. Determine $r'(-2)$ and $r'(0)$.

$r'(0) \approx 0 + 2 \cdot \frac{1}{2}$

$= \boxed{1}$

b. Are there values of x for which $r'(x)$ does not exist? If so, which values, and why? $x = -1$ or 1

p' & q' aren't defined there.

c. Find an equation for the tangent line to $y = r(x)$ at the point $(2, r(2))$.

$L(x) = 2 + 8(x-2)$

$r(2) = p(2)q(2)$

$= 1 \cdot 2 = 2$

d. Let $z(x) = \frac{q(x)}{p(x)}$. Determine $z'(0)$ and $z'(2)$. $\quad z'(x) = \frac{q'(x)p(x) - q(x)p'(x)}{(p(x))^2}$

$z'(0) = \frac{0 - 2\cdot\frac{1}{2}}{(-\frac{1}{2})^2} = \frac{-1}{1/4}$

$= -1$

e. Are there values of x for which $z'(x)$ does not exist? If so, which values, and why?

when p' & q' fail to exist, or when $g(x) = 0$.

$z'(2) = \frac{-1\cdot 2 - 1 \cdot 2}{(2)^2}$

$= -\frac{4}{4} = -1$

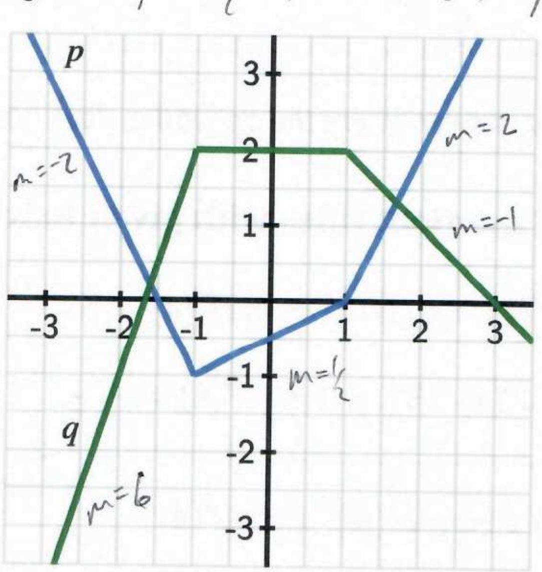


Figure 2.3.6. The graphs of p (in blue) and q (in green).

13. A farmer with large land holdings has historically grown a wide variety of crops. With the price of ethanol fuel rising, he decides that it would be prudent to devote more and more of his acreage to producing corn. As he grows more and more corn, he learns efficiencies that increase his yield per acre. In the present year, he used 7000 acres of his land to grow corn, and that land had an average yield of 170 bushels per acre. At the current time, he plans to increase his number of acres devoted to growing corn at a rate of 600 acres/year, and he expects that right now his average yield is increasing at a rate of 8 bushels per acre per year. Use this information to answer the following questions.