derivatives we know, and list the derivatives. Then write a composite function with the inner function being an unknown function $u(x)$ and the outer function being a basic function. Finally, write the chain rule for the composite function. The following example illustrates this for two different functions.

Example 2.5.8. To determine

$$\frac{d}{dx}[\sin(u(x))],$$

where u is a differentiable function of x, we use the chain rule with the sine function as the outer function. Applying the chain rule, we find that

$$\frac{d}{dx}[\sin(u(x))] = \cos(u(x)) \cdot u'(x).$$

This rule is analogous to the basic derivative rule that $\frac{d}{dx}[\sin(x)] = \cos(x)$.

Similarly, since $\frac{d}{dx}[a^x] = a^x \ln(a)$, it follows by the chain rule that

$$\frac{d}{dx}[a^{u(x)}] = a^{u(x)} \ln(a) \cdot u'(x).$$

This rule is analogous to the basic derivative rule that $\frac{d}{dx}[a^x] = a^x \ln(a)$.

stuff $= 5x$

stuff$' = 5$

$p(x) = e^{5x}$

$= e^{\text{stuff}}$

$p'(x) = e^{\text{stuff}} \cdot \text{stuff}'$

$= e^{5x} \cdot 5$

2.5.4 Summary

- A composite function is one where the input variable x first passes through one function, and then the resulting output passes through another. For example, the function $h(x) = 2^{\sin(x)}$ is composite since $x \mapsto \sin(x) \mapsto 2^{\sin(x)}$.
- Given a composite function $C(x) = f(g(x))$ where f and g are differentiable functions, the chain rule tells us that

$$C'(x) = f'(g(x))g'(x).$$

$f(x) = p(x) \cdot q(x)$ where

$p(x) = e^{5x}$

$q(x) = x^2 + 7^x$

$q'(x) = (x^2)' + (7^x)'$

$= 2x + \ln(7) \cdot 7^x$

2.5.5 Exercises

1. Mixing rules: chain, product, sum.

Activate

Find the derivative of

$f(x) = e^{5x}(x^2 + 7^x)$

$f'(x) = e^{5x}\left[5x^2 + 2x + (5 + \ln(7))\, 7^x\right]$

$\therefore f'(x) = p'(x)\, q(x) + p(x) \cdot q'(x)$

$= 5e^{5x}(x^2 + 7^x) + e^{5x}(2x + \ln(7) \cdot 7^x)$

$= e^{5x}(5x^2 + 5 \cdot 7^x + 2x + \ln(7) \cdot 7^x)$

2. Mixing rules: chain and product.

Activate

Find the derivative of

$v(t) = t^6 e^{-ct}$ (product) $p(t) = t^6 \quad q(t) = e^{-ct}$

General rule: $\frac{d}{dx}\left(e^{u(x)}\right) = u'(x) \cdot e^{u(x)}$

$\therefore \left(e^{-ct}\right)' = (-c)\, e^{-ct} = q'(t)$

$p'(t) = 6t^5$

Assume that c is a constant.

$v'(t) = (6t^5)e^{-ct} + t^6(-ce^{-ct})$

$= \boxed{t^5 e^{-ct}(6 - ct)}$

3. Using the chain rule repeatedly.

Activate

Find the derivative of see below

$y = \sqrt{e^{-5t^2} + 9}$

$\frac{dy}{dt} =$ ____________________

4. Derivative involving arbitrary constants a and b.

Activate

$p'(x) = a \quad q'(x) = (-bx+12)' e^{-bx+12} = -b e^{-bx+12}$

Find the derivative of

$f(x) = \underbrace{ax}_{p(x)}\underbrace{e^{-bx+12}}_{q(x)}$ product $= a \cdot e^{-bx+12} + ax(-be^{-bx+12})$

Assume that a and b are constants. $= \boxed{ae^{-bx+12}(1 - abx)}$

$f'(x) =$ ____________________

5. Chain rule with graphs.

$(\sqrt{x})' = \frac{1}{2\sqrt{x}}$

#3 $y = \sqrt{\text{stuff}}$

$y' = \frac{1}{2\sqrt{\text{stuff}}} \cdot \text{stuff}'$

$= \frac{1}{2\sqrt{e^{-5t^2}+9}} \left(e^{-5t^2} + 9\right)'$ sum rule

$= \frac{1}{2\sqrt{e^{-5t^2}+9}} \left(e^{-5t^2}\right)'$

$= \frac{1}{2\sqrt{e^{-5t^2}+9}} \cdot \left(-10t e^{-5t^2}\right)$

$= \boxed{\frac{-5t}{\sqrt{e^{-5t^2}+9}} \cdot e^{-5t^2}}$

$\left(e^{-5t^2}\right)' =$

$(-5t^2)' \cdot e^{-5t^2} =$

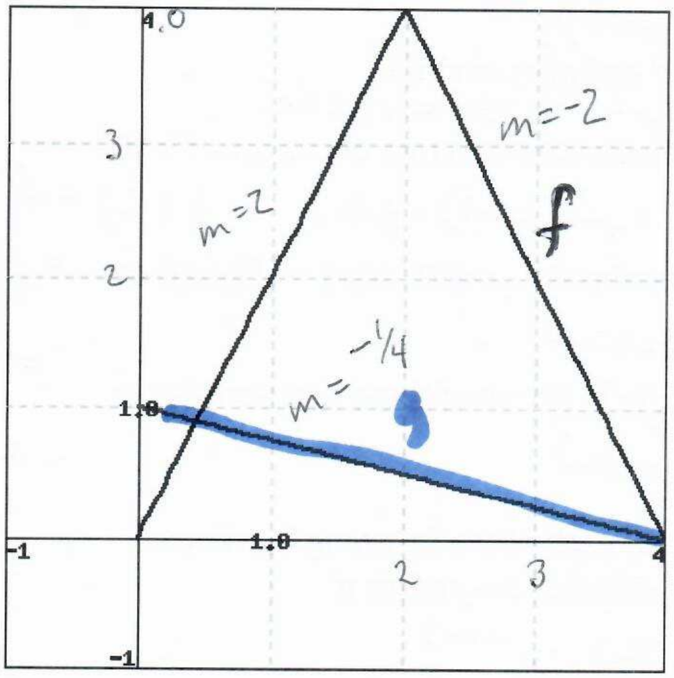
(see general rule above)

$(-5 \cdot 2t)\, e^{-5t^2} =$

$-10t e^{-5t^2}$

Activate

Use the graph below to find exact values of the indicated derivatives, or state that they do not exist. If a derivative does not exist, enter ***dne*** in the answer blank. The graph of $f(x)$ is black and has a sharp corner at $x = 2$. The graph of $g(x)$ is blue.



Let $h(x) = f(g(x))$. Find

A. $h'(1) =$ $f'(g(1)) \cdot g'(1) = f'(\frac{3}{4}) \cdot (-\frac{1}{4}) = 2(-\frac{1}{4}) = -\frac{1}{2}$

B. $h'(2) =$ $f'(g(2)) \cdot g'(2) = f'(\frac{1}{2}) \cdot (-\frac{1}{4}) = 2 \cdot (-\frac{1}{4}) = -\frac{1}{2}$

C. $h'(3) =$ $f'(g(3)) \cdot g'(3) = f'(\frac{1}{4}) \cdot (-\frac{1}{4}) = 2 \cdot (-\frac{1}{4}) = -\frac{1}{2}$

*(Enter **dne** for any derivative that does not exist.)*

6. Chain rule with function values.

Activate

Given $F(4) = 1, F'(4) = 5, F(5) = 4, F'(5) = 6$ and $G(1) = 3, G'(1) = 4, G(4) = 5, G'(4) = 6$, find each of the following. (Enter ***dne*** for any derivative that cannot be computed from this information alone.)

A. $H(4)$ if $H(x) = F(G(x))$ $H(4) = F(G(4)) = F(5) = \boxed{4}$

B. $H'(4)$ if $H(x) = F(G(x))$ $H'(4) = F'(G(4)) \cdot G'(4) = F'(5) \cdot 6 = 6 \cdot 6 = \boxed{36}$

C. $H(4)$ if $H(x) = G(F(x))$ $H(4) = G(F(4)) = G(1) = \boxed{3}$

D. $H'(4)$ if $H(x) = G(F(x))$ $H'(4) = G'(F(4)) \cdot F'(4) = G'(1) \cdot 5 = 4 \cdot 5 = \boxed{20}$

E. $H'(4)$ if $H(x) = F(x)/G(x)$ $H'(4) = \dfrac{G(4)F'(4) - G'(4)F(4)}{G(4)^2} = \dfrac{5 \cdot 5 - 6 \cdot 1}{5^2} = \boxed{\dfrac{19}{25}}$

Activate

Find the derivative of $f(x) = \underbrace{2x}_{p(x)} \underbrace{\sin(6x)}_{q(x)}$ product

$p'(x) = 2$ $\quad q'(x) = \cos(\text{stuff}) \cdot \text{stuff}'$

$= \cos(6x) \cdot (6)$

$= 6\cos(6x)$

$f'(x) = 2\sin(6x) + 2x \cdot 6\cos(6x)$

$= 2(\sin(6x) + 6x\cos(6x))$

8. Consider the basic functions $f(x) = x^3$ and $g(x) = \sin(x)$.

$f'(x) = 3x^2 \quad g'(x) = \cos(x)$

a. Let $h(x) = f(g(x))$. Find the exact instantaneous rate of change of h at the point where $x = \frac{\pi}{4}$.

$h'(x) = f'(g(x)) \cdot g'(x)$

$h'(\frac{\pi}{4}) = f'(g(\frac{\pi}{4})) \cdot g'(\frac{\pi}{4})$

$= f'(\frac{\sqrt{2}}{2}) \cdot \frac{\sqrt{2}}{2}$

$= 3(\frac{\sqrt{2}}{2})^2 \cdot \frac{\sqrt{2}}{2}$

$= \frac{3}{4}\sqrt{2}$

b. Which function is changing most rapidly at $x = 0.25$: $h(x) = f(g(x))$ or $r(x) = g(f(x))$? Why?

see below

c. Let $h(x) = f(g(x))$ and $r(x) = g(f(x))$. Which of these functions has a derivative that is periodic? Why?

see below

9. Let $u(x)$ be a differentiable function. For each of the following functions, determine the derivative. Each response will involve u and/or u'.

a. $p(x) = e^{u(x)}$ (stuff) $\quad p'(x) = u'(x) \cdot e^{u(x)}$

b. $q(x) = u(e^x)$ $\quad q'(x) = u'(e^x) \cdot (e^x)' = u'(e^x) \cdot e^x$

c. $r(x) = \cot(u(x))$ $\quad r'(x) = -\csc^2(u(x)) \cdot u'(x)$

d. $s(x) = u(\cot(x))$ $\quad s'(x) = u'(\cot(x)) \cdot (-\csc^2(x))$

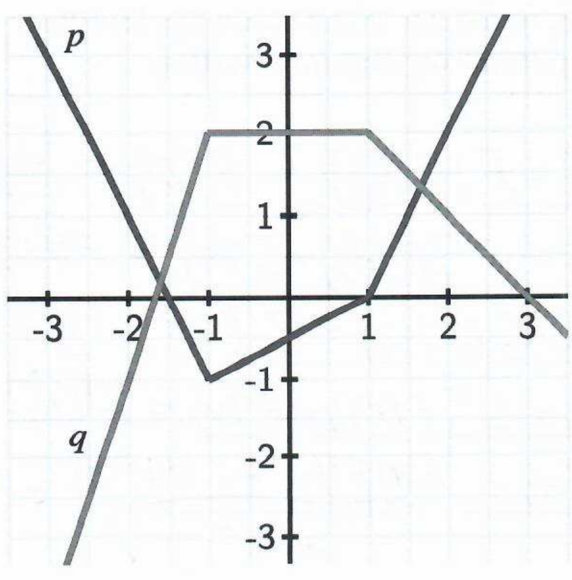
e. $a(x) = u(x^4)$ $\quad a'(x) = u'(x^4) \cdot (x^4)' = u'(x^4) \cdot 4x^3$

f. $b(x) = u^4(x) = (u(x))^4$ $\quad b'(x) = 4(u(x))^3 \cdot u'(x)$

stuff

10. Let functions p and q be the piecewise linear functions given by their respective graphs in Figure 2.5.9. Use the graphs to answer the following questions.

8c. $h(x) = (\sin(x))^3$ is periodic; $\sin(x^3)$ is not.



8b. $h'(x) = 3(\sin(x))^2 \cdot \cos(x)$

$r'(x) = g'(f(x)) \cdot f'(x)$

$= \cos(x^3) \cdot 3x^2$

$h'(.25) \approx .178$

$r'(.25) \approx .187$

r is changing more rapidly

Homework (Section 2.5): Chain Rule

Andy Long, Spring 2024

8. Consider the basic functions $f(x) = x^3$ and $g(x) = \sin(x)$.

a. Let $h(x) = f(g(x))$. Find the exact instantaneous rate of change of h at the point where $x = \frac{\pi}{4}$.

b. Which function is changing most rapidly at $x = 0.25$: $h(x) = f(g(x))$ or $r(x) = g(f(x))$? Why?

c. Let $h(x) = f(g(x))$ and $r(x) = g(f(x))$. Which of these functions has a derivative that is periodic? Why?

a.

In[3225]:=
```
Clear[x]
f[x_] := x^3
g[x_] := Sin[x]
h[x_] := f[g[x]]
h'[x]
h'[Pi / 4]
```

Out[3229]= $3\,\mathrm{Cos}[x]\,\mathrm{Sin}[x]^2$

Out[3230]= $\frac{3}{2\sqrt{2}}$

b.

In[3231]:=
```
h'[x]
r[x_] := g[f[x]]
r'[x]
h'[.25]
r'[.25]
```

Out[3231]= $3\,\mathrm{Cos}[x]\,\mathrm{Sin}[x]^2$

Out[3233]= $3\,x^2\,\mathrm{Cos}[x^3]$

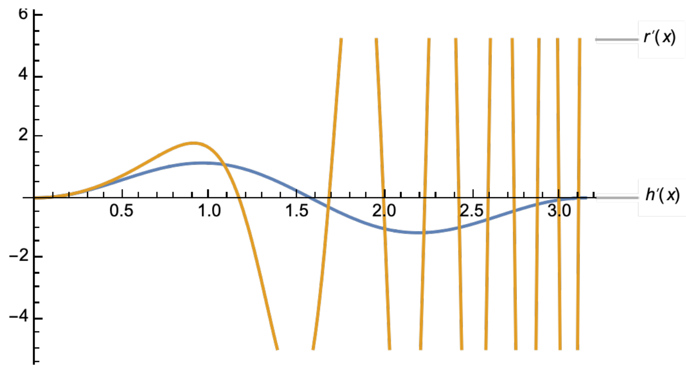
Out[3234]= 0.177917664627618

Out[3235]= 0.187477112282064

c. h’(x) is periodic....

In[3238]:=
```
Plot[{h'[x], r'[x]}, {x, 0, Pi}, PlotLabels → Automatic]
```

Out[3238]=



11. If a spherical tank of radius 4 feet has h feet of water present in the tank, then the volume of water in the tank is given by the formula

$$V = \frac{\pi}{3}h^2(12 - h).$$

a. At what instantaneous rate is the volume of water in the tank changing with respect to the *height* of the water at the instant $h = 1$? What are the units on this quantity?

b. Now suppose that the height of water in the tank is being regulated by an inflow and outflow (e.g., a faucet and a drain) so that the height of the water at time t is given by the rule $h(t) = \sin(\pi t) + 1$, where t is measured in hours (and h is still measured in feet). At what rate is the height of the water changing with respect to time at the instant $t = 2$?

c. Continuing under the assumptions in (b), at what instantaneous rate is the volume of water in the tank changing with respect to *time* at the instant $t = 2$?

d. What are the main differences between the rates found in (a) and (c)? Include a discussion of the relevant units.

---

a.

In[3214]:=
```
V[h_] := Pi / 3 h^2 (12 - h)
Plot[V[h], {h, 0, 8}, AxesLabel → {"height", "Volume (cubic feet)"}]
V'[h]
V'[1.0]
(* with units feet-squared (volume divided by length) *)
```

b.

In[3221]:=
```
h[t_] := Sin[Pi t] + 1
Plot[h[t], {t, 0, 3}, AxesLabel → {"time (hours)", "height (feet)"}]
h'[t]
h'[2.0]
```

c.

```
V'[h[t]] × h'[t]
Plot[V[h[t]], {t, 0, 3}, AxesLabel → {"time (hours)", "Volume (cubic feet)"}]
V'[h[2.0]] × h'[2.0]
```

d. The rate in part (a) is the rate of change of Volume with respect to height... (ft)^3/ft = ft^2; when we have the height as a function of time, we can ask what is the rate of change of Volume with respect to time.... (ft)^3/hour