Section 3.3 Homework

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Problems 1-4

1. Based on the given information about each function, decide whether the function has global maximum, a global minimum, neither, both, or that it is not possible to say without more information. Assume that each function is twice differentiable and defined for all real numbers, unless noted otherwise. In each case, write one sentence to explain your conclusion.

a. f is a function such that $f''(x) < 0$ for every x.

b. g is a function with two critical numbers a and b (where $a < b$), and $g'(x) < 0$ for $x < a$, $g'(x) < 0$ for $a < x < b$, and $g'(x) > 0$ for $x > b$.

c. h is a function with two critical numbers a and b (where $a < b$), and $h'(x) < 0$ for $x < a$, $h'(x) > 0$ for $a < x < b$, and $h'(x) < 0$ for $x > b$. In addition, $\lim_{x \to \infty} h(x) = 0$ and $\lim_{x \to -\infty} h(x) = 0$.

d. p is a function differentiable everywhere except at $x = a$ and $p''(x) > 0$ for $x < a$ and $p''(x) < 0$ for $x > a$.

1a. This function is like -x^2, so it has a global maximum, and no minimum. It is concave down everywhere.

1b. There is a global minimum at b, and an inflection point at a; there is no global maximum.

1c. We can't say if there is a global minimum at a; there is at least a local minimum. Similarly there is at least a local maximum at b, but it may not be a global maximum.

1d. We don't know what's going on in terms of a, or global extrema; It's concave up up to a, and concave down to the right of a. That's all we can say. There may be a dramatic discontinuity at x=a.

2. For each family of functions that depends on one or more parameters, determine the function's absolute maximum and absolute minimum on the given interval.

a. $p(x) = x^3 - a^2 x$, $[0, a]$ $(a > 0)$

b. $r(x) = axe^{-bx}$, $[\frac{1}{2b}, \frac{2}{b}]$ $(a > 0, b > 1)$

c. $w(x) = a(1 - e^{-bx})$, $[b, 3b]$ $(a, b > 0)$

d. $s(x) = \sin(kx)$, $\left[\frac{\pi}{3k}, \frac{5\pi}{6k}\right]$ $(k > 0)$

2a. This is a cubic, with zeros at 0, and zeros at + and - a. Since the lead coefficient is positive, this cubic goes to infinity as x goes to infinity. So there is a max at -a/Sqrt[3], and a min at a/Sqrt[3]

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In[763]:= Clear[a, b, k]
p[x_] := x^3 - a^2 x;
solns = Solve[p'[x] == 0, x]
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Out[765]= $\left\{\left\{x \to -\frac{a}{\sqrt{3}}\right\}, \left\{x \to \frac{a}{\sqrt{3}}\right\}\right\}$

2b. This is a linear function with positive slope which multiplies a dying exponential. So it's going to rise from 0, become positive, but eventually die back to the x-axis. The a does nothing but scale the function up or down -- it doesn't affect extrema. There is a max at x=1/b.

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In[766]:= r[x_] := a x E^(-b x);
solns = Solve[r'[x] == 0, x]
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Out[767]= $\left\{\left\{x \to \frac{1}{b}\right\}\right\}$

2c. Since the function is equal at its endpoints, and is concave down, there is an absolute maximum between the endpoints, and the endpoints are global minima.

2d. on [2Pi/(6k), 5Pi/(6k)] ; but when these values of x are used, s[x] is equal to Sin[x] on the interval [2Pi/6,5Pi/6]. And there's a global max at Pi/2 (meaning Pi/(2k) in general).

In[768]:=
```
s[x_] := Sin[k x];
Plot[Sin[x], {x, 2 Pi / 6, 5 Pi / 6}]
solns = Solve[Sin'[x] == 0, x]
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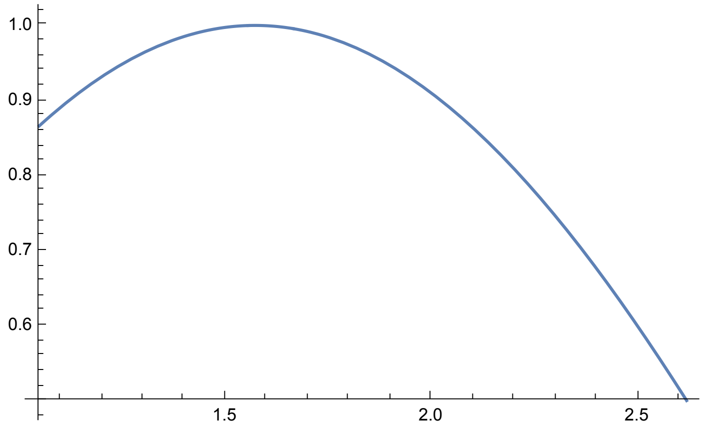
Out[769]=



Out[770]= $\left\{\left\{x \to -\frac{\pi}{2} + 2\pi \mathbb{c}_1 \text{ if } \mathbb{c}_1 \in \mathbb{Z}\right\}, \left\{x \to \frac{\pi}{2} + 2\pi \mathbb{c}_1 \text{ if } \mathbb{c}_1 \in \mathbb{Z}\right\}\right\}$

3. For each of the functions described below (each continuous on $[a, b]$), state the location of the function's absolute maximum and absolute minimum on the interval $[a, b]$, or say there is not enough information provided to make a conclusion. Assume that any critical numbers mentioned in the problem statement represent all of the critical numbers the function has in $[a, b]$. In each case, write one sentence to explain your answer.

a. $f'(x) \leq 0$ for all x in $[a, b]$

b. g has a critical number at c such that $a < c < b$ and $g'(x) > 0$ for $x < c$ and $g'(x) < 0$ for $x > c$

c. $h(a) = h(b)$ and $h''(x) < 0$ for all x in $[a, b]$

d. $p(a) > 0$, $p(b) < 0$, and for the critical number c such that $a < c < b$, $p'(x) < 0$ for $x < c$ and $p'(x) > 0$ for $x > c$

3a. Since the derivative could be identically zero, a constant function is a possibility. Then every point would be a global maximum and global minimum.

3b. There is a global maximum at c, by the first derivative test and extreme value theorem.

3c. There is a single global maximum between a and b.

3d. There is a global minimum x=c because it drops from the positive side, to below the right endpoint, and then climbs back up to the right endpoint

4. Let $s(t) = 3\sin(2(t - \frac{\pi}{6})) + 5$. Find the exact absolute maximum and minimum of s on the provided intervals by testing the endpoints and finding and evaluating all relevant critical numbers of s.

a. $[\frac{\pi}{6}, \frac{7\pi}{6}]$

b. $[0, \frac{\pi}{2}]$

c. $[0, 2\pi]$

d. $[\frac{\pi}{3}, \frac{5\pi}{6}]$

In[771]:=
```
s[t_] := 3 Sin[ 2 (t - Pi / 6)] + 5;
s'[t]
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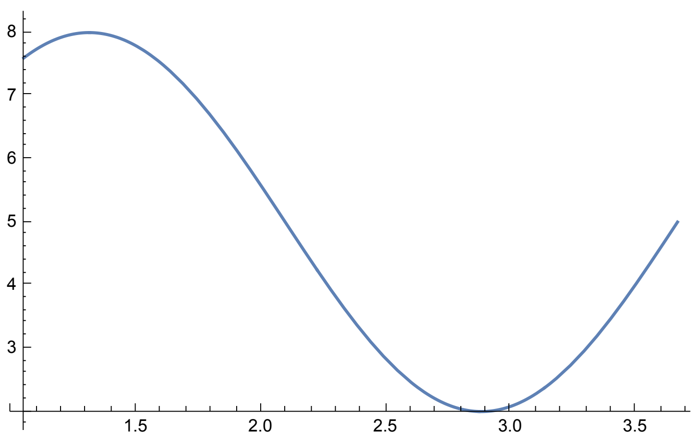
Out[772]= $6\,\mathrm{Cos}\left[2\left(-\frac{\pi}{6}+t\right)\right]$

4a.

In[773]:=
```
{a, b} = {2 Pi / 6, 7 Pi / 6};
Plot[s[x], {x, a, b}]
solns = Solve[s'[x] == 0, x, Assumptions → a ≤ x ≤ b]
```

Out[774]=

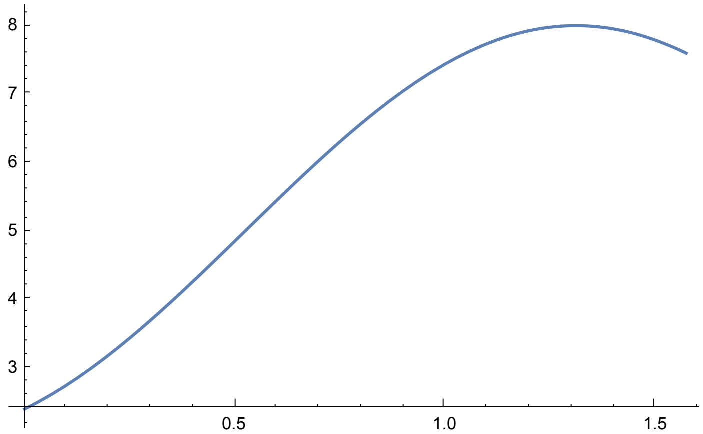


Out[775]= $\left\{\left\{x \to \frac{5\pi}{12}\right\}, \left\{x \to \frac{11\pi}{12}\right\}\right\}$

4b.

In[776]:=
```
{a, b} = {0, Pi / 2};
Plot[s[x], {x, a, b}]
solns = Solve[s'[x] == 0, x, Assumptions → a ≤ x ≤ b]
```

Out[777]=



Out[778]= $\left\{\left\{x \to \frac{5\pi}{12}\right\}\right\}$

4c.

In[779]:=
```
{a, b} = {0, 2 Pi};
Plot[s[x], {x, a, b}]
solns = Solve[s'[x] == 0, x, Assumptions → a ≤ x ≤ b]
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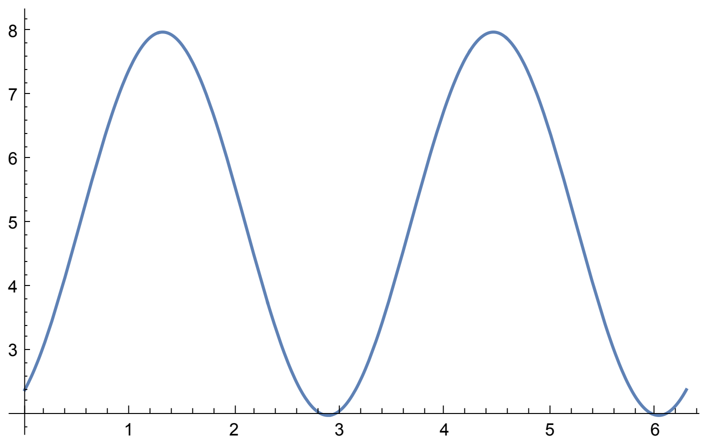
Out[780]=



Out[781]= $\left\{\left\{x \to \frac{5\pi}{12}\right\}, \left\{x \to \frac{11\pi}{12}\right\}, \left\{x \to \frac{17\pi}{12}\right\}, \left\{x \to \frac{23\pi}{12}\right\}\right\}$

4d.

In[782]:=
```
{a, b} = {Pi / 3, 5 Pi / 6};
Plot[s[x], {x, a, b}]
solns = Solve[s'[x] == 0, x, Assumptions → a ≤ x ≤ b]
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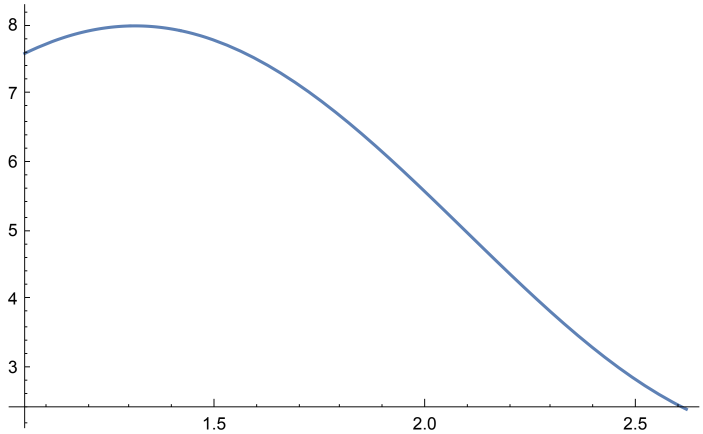
Out[783]=



Out[784]= $\left\{\left\{x \to \frac{5\pi}{12}\right\}\right\}$