Let $F(x) = c$ a constant.

~~Let~~

$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h}$$

$$= \lim_{h \to 0} \frac{c - c}{h}$$

$$= \lim_{h \to 0} \frac{0}{h} = \lim_{h \to 0} 0 = 0$$

$$\frac{d}{dx}(c) = 0$$

Let $F(x) = x$.

$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h) - x}{h}$$

$$= \lim_{h \to 0} \frac{h}{h} = \lim_{h \to 0} 1 = 1$$

$$\frac{d}{dx}(x) = 1$$

Let $F(x) = f(x) + g(x)$, both differentiable at x

$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h}$$

$$= \lim_{h \to 0} \frac{[f(x+h) + g(x+h)] - [f(x) + g(x)]}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h) + g(x+h) - f(x) - g(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h) - f(x) + g(x+h) - g(x)}{h}$$

$$= \lim_{h \to 0} \left[\frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h} \right]$$

Limit of a sum is the sum of the limits

$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

$$= f'(x) + g'(x)$$

$$(f(x) + g(x))' = f'(x) + g'(x)$$

(Sum Rule)

Let $F(x) = f(x)\cdot g(x)$ both differentiable at x

$$F'(x) = \lim_{h\to 0} \frac{F(x+h) - F(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)\cdot g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) \overbrace{- f(x)g(x+h) + f(x)g(x+h)}^{=0} - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \frac{[f(x+h) - f(x)]\,g(x+h) + f(x)[g(x+h) - g(x)]}{h}$$

$$= \lim_{h\to 0} \left[\frac{[f(x+h) - f(x)]}{h} g(x+h) + \frac{f(x)[g(x+h) - g(x)]}{h} \right]$$

$$= \lim_{h\to 0} \left[\frac{(f(x+h) - f(x))}{h} g(x+h) \right] + \lim_{h\to 0} \left(\frac{f(x)[g(x+h) - g(x)]}{h} \right)$$

$$= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \cdot \lim_{h\to 0} g(x+h) + f(x) \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}$$

$$= g(x) \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} + f(x) \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}$$

$$= g(x)\, f'(x) + f(x)\cdot g'(x)$$

$$= f'(x)\, g(x) + f(x)\, g'(x)$$

product rule:

$$\boxed{(f(x)\cdot g(x))' = f'(x)g(x) + f(x)g'(x)}$$

Corollaries: Note: no limit definition!

Constant Multiple rule

$F(x) = c f(x)$ c is constant

(a product!)

$F'(x) = (c)' f(x) + c f'(x)$ using the product rule

$= 0 \cdot f(x) + c f'(x)$

$(c f(x))' = \boxed{c \cdot f'(x)}$

Difference rule:

$F(x) = f(x) - g(x)$

$F'(x) = [f(x) + (-g(x))]'$

$= [f(x) + (-1) \cdot g(x)]'$

$= f'(x) + (-1) g'(x)$

$(f(x) - g(x))' = f'(x) - g'(x)$

Let $F(x) = x^n$. Then $F'(x) = nx^{n-1}$.

<u>Base case</u> (anchor): $n=1$

$$F(x) = x^1 = x$$
$$F'(x) = 1 = 1\cdot x^{1-1} = 1\cdot x^0 \checkmark$$

<u>Induction step</u>: <u>Assume</u> the rule works for the k^{th} case:

$$F(x) = x^k, \text{ then } F'(x) = kx^{k-1}$$

and consider

$$x^{k+1}:$$

$$(x^{k+1})' = (x^k \cdot x)'$$
$$= (x^k)' \cdot x + x^k (x)'$$

By the product rule!

$$= kx^{k-1} \cdot x + x^k \cdot (1)$$
$$= kx^k + x^k$$
$$= (k+1)x^k = (k+1)x^{(k+1)-1} \checkmark$$