Let $F(x) = \dfrac{f(x)}{g(x)}$ $\quad f + g$ both differentiable at x ; $g(x) \neq 0$.

$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h} = \lim_{h \to 0} \frac{1}{h}\left[\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{f(x+h)\,g(x) - f(x)\,g(x+h)}{g(x+h)\,g(x)}\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{f(x+h)\,g(x) - f(x)g(x) + f(x)g(x) - f(x)\,g(x+h)}{g(x+h)\,g(x)}\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{[f(x+h) - f(x)]\,g(x) + f(x)\,[g(x) - g(x+h)]}{g(x+h)\,g(x)}\right]$$

$$= \lim_{h \to 0}\left[\frac{\frac{f(x+h) - f(x)}{h}\,g(x) + f(x)\,\frac{g(x) - g(x+h)}{h}}{g(x+h)\,g(x)}\right]$$

$$= \frac{\lim_{h \to 0}\left[\frac{f(x+h) - f(x)}{h} \cdot g(x) - f(x)\,\frac{g(x+h) - g(x)}{h}\right]}{\lim_{h \to 0}\left[g(x+h)\,g(x)\right]}$$

$$= \frac{f'(x)\,g(x) - f(x)\,g'(x)}{g(x)^2}$$

A different way...

Let $F(x) = \dfrac{1}{g(x)}$ $\quad$ g is diff. at x & $g(x) \neq 0$

What's

$$\left[g(x) \cdot F(x)\right]' \; ?$$

$$g(x) \cdot F(x) = g(x) \cdot \frac{1}{g(x)} = 1$$

So $\left[g(x) \cdot F(x)\right]' = 0$

$$\left[g(x) \cdot F(x)\right]' =$$

$$g'(x) \cdot F(x) + g(x) \cdot F'(x) = 0$$

Solve for $F'(x)$:

$$F'(x) = -\frac{g'(x) F(x)}{g(x)} = \frac{-g'(x) \frac{1}{g(x)}}{g(x)}$$

$$\boxed{F'(x) = \frac{-g'(x)}{g(x)^2}}$$

Consider

$$\left(\frac{f(x)}{g(x)}\right)' = \left(f(x)\cdot\frac{1}{g(x)}\right)'$$

$$= f'(x)\cdot\frac{1}{g(x)} + f(x)\left(\frac{1}{g(x)}\right)'$$

$$= \frac{f'(x)}{g(x)} + f(x)\left(-\frac{g'(x)}{g(x)^2}\right)$$

$$= \frac{f'(x)g(x)}{g(x)^2} - \frac{f(x)\cdot g'(x)}{g(x)^2}$$

$$= \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$ The Quotient Rule.