Dogs Do Calculus

Andy Long, MAT128
Spring, 2024

Based on the article "Do Dogs Know Calculus?", by Tim Pennings :

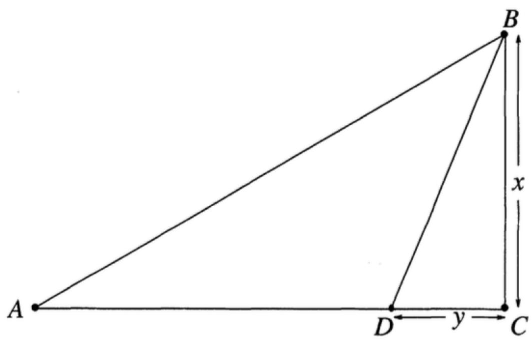


Figure 1. Paths to the ball

Let r denote the running speed, and s be the swimming speed. (Our units will be meters and seconds.) Let $T(y)$ represent the time to get to the ball given that Elvis jumps into the water at D, which is y meters from C. Let z represent the entire distance from A to C. Since time = distance/speed, we have

$$T(y) = \frac{z-y}{r} + \frac{\sqrt{x^2+y^2}}{s}. \qquad (1)$$

We want to find the value of y that minimizes $T(y)$. Of course this happens where $T'(y) = 0$. Solving $T'(y) = 0$ for y, we get

$$y = \frac{x}{\sqrt{r/s+1}\,\sqrt{r/s-1}}, \qquad (2)$$

So the theoretical relationship between y and x is purely linear, with a slope of

In[1019]:=
```
Clear[r, s, m, g]
slope = 1 / Sqrt[(r / s) ^2 - 1]
```

Out[1020]= $\frac{1}{\sqrt{-1+\frac{r^2}{s^2}}}$

In[1021]:=
```
time[y_] := (z - y) / r + Sqrt[x^2 + y^2] / s
time'[y]
solns = Solve[time'[y] == 0, y]
```

Out[1022]= $-\frac{1}{r} + \frac{y}{s\sqrt{x^2+y^2}}$

Out[1023]= $\left\{\left\{y \to -\frac{s\,x}{\sqrt{r^2-s^2}}\right\}, \left\{y \to \frac{s\,x}{\sqrt{r^2-s^2}}\right\}\right\}$

I think that we can agree that the negative answer probably doesn’t make sense....

The data is included in the article as a table: I've transcribed it here:

Table 2. Throw and fetch trials

x	y	x	y	x	y	x	y	x	y
10.5	2.0	17.0	2.1	4.7	0.9	10.9	2.2	15.3	2.3
7.2	1.0	15.6	3.9	11.6	2.2	11.2	1.3	11.8	2.2
10.3	1.8	6.6	1.0	11.5	1.8	15.0	3.8	7.5	1.4
11.7	1.5	14.0	2.6	9.2	1.7	14.5	1.9	11.5	2.1
12.2	2.3	13.4	1.5	13.5	1.8	6.0	0.9	12.7	2.3
19.2	4.2	6.5	1.0	14.2	1.9	14.5	2.0	6.6	0.8
11.4	1.3	11.8	2.4	14.2	2.5	12.5	1.5	15.3	3.3

```
In[1024]:= x = {10.5, 17.0, 4.7, 10.9, 15.3, 7.2, 15.6, 11.6, 11.2,
    11.8, 10.3, 6.6, 11.5, 15.0, 7.5, 11.7, 14.0, 9.2, 14.5, 11.5, 12.2, 13.4,
    13.5, 6.0, 12.7, 19.2, 6.5, 14.2, 14.5, 6.6, 11.4, 11.8, 14.2, 12.5, 15.3};
y = {2.0, 2.1, 0.9, 2.2, 2.3, 1.0, 3.9, 2.2, 1.3, 2.2, 1.8, 1.0, 1.8, 3.8, 1.4, 1.5, 2.6, 1.7,
    1.9, 2.1, 2.3, 1.5, 1.8, 0.9, 2.3, 4.2, 1.0, 1.9, 2.0, 0.8, 1.3, 2.4, 2.5, 1.5, 3.3};
data = Transpose[{x, y}];
```

"I also omitted the couple of times when Elvis, in his haste and excitement, jumped immediately into the water and swam the entire distance. I figured that even an 'A' student can have a bad day."

We'll need Elvis's parameters for speeds, on land and in the water:

Table 1. Running and swimming times

Running times (in seconds) for 20 meters	Swimming times (in seconds) for 10 meters
3.20	12.13
3.16	11.15
3.15	11.07
3.13	10.75
3.10	12.22

These give average speeds of (r for running, s for swimming):

```
In[1027]:= r = Mean[20 / {3.20, 3.16, 3.15, 3.13, 3.10}]
s = Mean[10 / {12.13, 11.15, 11.07, 10.75, 12.22}]
slope

Out[1027]= 6.35394190686205

Out[1028]= 0.874633765066946

Out[1029]= 0.1389751050391
```

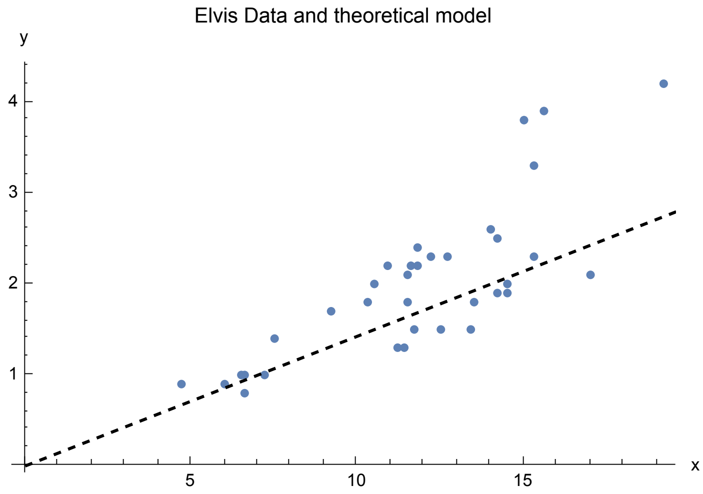
(although Timothy took the average of only the top three speeds in his analysis).

I might argue for **median** speeds, rather than **means**:

```
In[1030]:= r = Median[20 / {3.20, 3.16, 3.15, 3.13, 3.10}]
s = Median[10 / {12.13, 11.15, 11.07, 10.75, 12.22}]
slope

Out[1030]= 6.34920634920635

Out[1031]= 0.896860986547085

Out[1032]= 0.142686298322051
```

Let's look at some plots, of data versus the theoretical model from the optimization model of calculus:

In[1033]:=
```
dataplot = ListPlot[data, AxesOrigin → {0, 0}, PlotLabel → "Elvis Data and theoretical model",
   AxesLabel → {"x", "y"}, AxesLabel → Automatic];
theoreticalmodelplot = Plot[slope x, {x, 0, 20}, PlotStyle → {Black, Dashed}];
Show[{dataplot, theoreticalmodelplot}]
```

Out[1035]=


Here we use linear regression to find the optimal slope, that minimizes the sum of squares.

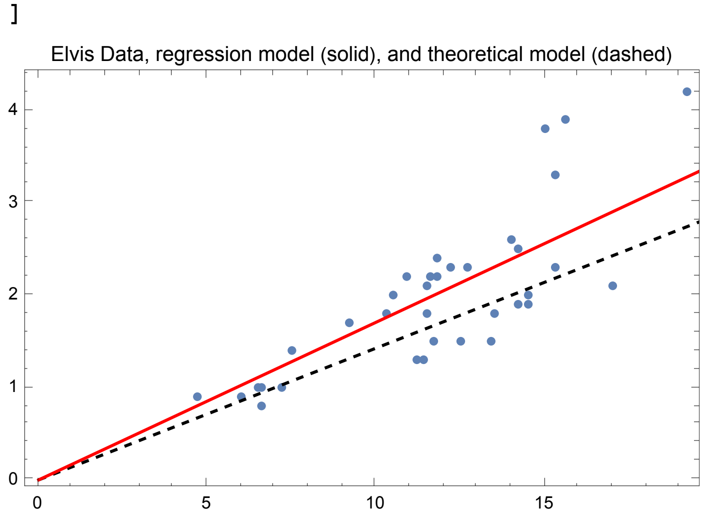
In[1036]:=
```
lm = LinearModelFit[data, t, t, IncludeConstantBasis → False]
regressionmodelplot =
  Plot[lm[x], {x, 0, 20}, PlotStyle → Red, AxesLabel → {"x", "y"}, AxesLabel → Automatic];
lm["ParameterTable"]
```

Out[1036]= FittedModel[0.170672543483085 t]

Out[1038]=

	Estimate	Standard Error	t-Statistic	P-Value
t	0.170672543483085	0.00714481970803254	23.8875927535592	$7.84935548538768\times10^{-23}$

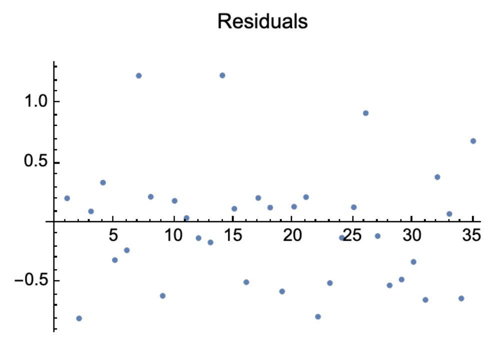
In[1039]:=
```
Show[
 dataplot,
 theoreticalmodelplot,
 regressionmodelplot,
 Frame → True,
 PlotLabel → "Elvis Data, regression model (solid), and theoretical model (dashed)"
]
```

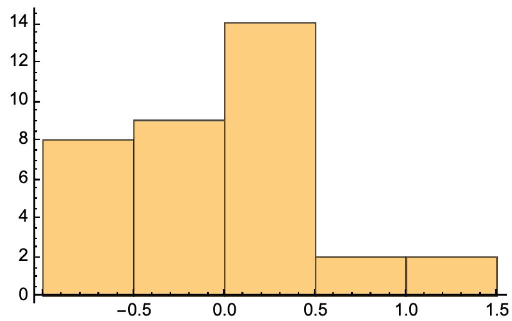
Out[1039]=


Here we use calculus to find the slope -- that’s what linear regression does, too! And we get the same answer, of course:

```
In[1040]:= f[m_] := Sum[(y[[i]] - m * x[[i]])^2, {i, 1, 35}]
           Solve[f'[m] == 0, m]

Out[1041]= {{m → 0.170672543483085}}
```

Out[1042]=





To my (maybe biased) eye, the agreement looks good. It seems clear that in most cases Elvis chose a path that agreed remarkably closely with the optimal path. The way to rigorously validate (and quantify) what the scatter plot suggests is to do a statistical analysis of the data. We will not do this in this paper, but it would be a natural avenue for further work. We conclude with several pertinent points.

First, we are in fact using a mathematical model. That is to say, we arrived at our theoretical figure by making many simplifying assumptions. These include

- We assumed there was a definite line between shore and lake. Because of waves, this was not the case.
- We assumed that when Elvis entered the water, he started swimming. Actually, he ran a short distance in the water. (Although given his six-inch legs, this is not too bad of an assumption!)
- We assumed the ball was stationary in the water. Actually, the waves, winds, and currents moved it a slight distance between the time Elvis plunged into the water and when he grabbed it.
- We assumed that the values of r and s are constant, independent of the distance run or swum.

Given these complicating factors as well as the error in measurements, it is possible that Elvis chose paths that were actually *better* than the calculated ideal path.

Second, we confess that although he made good choices, Elvis does not know calculus. In fact, he has trouble differentiating even simple polynomials. More seriously, although he does not do the calculations, Elvis's behavior is an example of the uncanny way in which nature (or Nature) often finds optimal solutions. Consider how soap bubbles minimize surface area, for example. It is fascinating that this optimizing ability seems to extend even to animal behavior. (It could be a consequence of natural selection, which gives a slight but consequential advantage to those animals that exhibit better judgment.)

Finally, for those intrigued by this general study, there are further experiments that are available, other than using your own favorite dog. One might do a similar experiment with a dog running in deep snow versus a cleared sidewalk. Even more interesting, one might test to determine whether the optimal path is found by six-year-old children, junior high aged pupils, or college students. For the sake of their pride, it might be best not to include professors in the study.