Directions: Show our work: answers without justification will likely result in few points. Your written work also allows me the option of giving you partial credit in the event of an incorrect final answer (but good reasoning). Indicate clearly your answers to each problem (e.g., put a box around them); and clearly separate solutions to each problem from other problems. **Good luck!**

Problem 1: (20 pts) Given the two functions

$$f(x) = x^3 \qquad\qquad g(x) = \ln(x)$$

a. (16 pts) Demonstrate your understanding of the chain rule to compute these derivatives:

i. $p(x) = f(g(x))$

$P(x) = f(g(x)) = (\ln(x))^3$

$p(x) = (\ln(x))^3$

$p'(x) = 3(\ln(x))^2 \cdot \frac{1}{x}$

$f'(g(x)) \cdot g'(x)$

$f(x) = x^3$

$f'(x) = 3x^2$

$g(x) = \ln(x)$

$g'(x) = \frac{1}{x}$

ii. $q(x) = g(f(x))$

$q(x) = g(f(x)) = \ln(x^3)$

$q(x) = \ln(x^3)$

$q'(x) = \frac{1}{x^3} \cdot 3x^2$

$= \frac{3x^2}{x^3}$

$q'(x) = \frac{3}{x}$

$f'(g(x)) \cdot g'(x)$

$f(x) = \ln(x)$

$f'(x) = \frac{1}{x}$

$g(x) = x^3$

$g'(x) = 3x^2$

b. (4 pts) We used the limit definition of the derivative to discover the chain rule, and here were the first two steps:

$$\lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{h} \approx \lim_{h\to 0} \frac{f(g(x) + hg'(x)) - f(g(x))}{h}$$

Explain why that approximation is reasonable.

It is reasonable because after that you can multiply by the right form of ~~zero~~ one.

Yes, that's the next step. But why can we take this step?

-2

18

b. (4 pts) We used the limit definition of the derivative to discover the chain rule, and here were the first two steps:

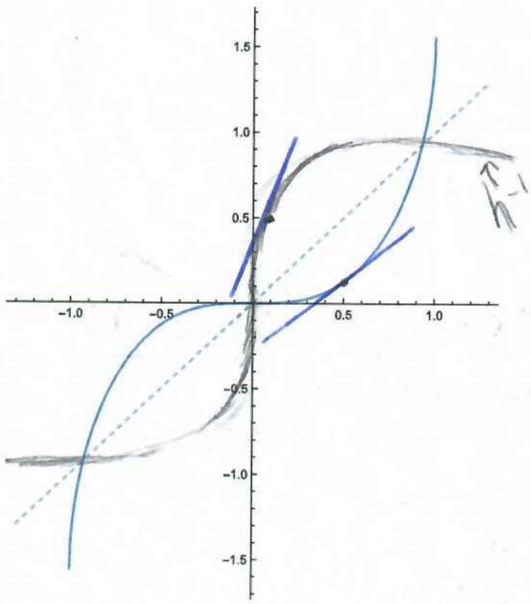
$$\lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{h} \approx \lim_{h\to 0} \frac{f(g(x) + hg'(x)) - f(g(x))}{h}$$

Explain why that approximation is reasonable.

$g(x+h) \simeq g(x) + hg'(x)$

is the local linearization at $\underline{x}$.

Problem 2: (20 pts) The graph of an invertible function, $h(x) = \arcsin(x^3)$, is given below:



a. (4pts) **Carefully** draw the inverse function h^{-1} into the graph.

b. (6pts) Find the derivative of $h(x)$ at $(0.5, 0.1253)$

$h'(x) = \frac{1}{\sqrt{1-(x^3)^2}} \cdot 3x^2 = \frac{1}{\sqrt{1-x^6}} \cdot 3x^2$

$\frac{1}{\sqrt{1-(.5)^6}} \cdot 3(.5)^2 = \boxed{0.7559}$

c. (5 pts) Draw the tangent line at $x = 0.5$ onto the graph of h, and find its equation.

$y - 0.1253 = 0.7559(x - 0.5)$

d. (5 pts) Draw the tangent line to the graph of the inverse function at $y = 0.5$, and write its equation (you don't need to do any additional calculations!). What do you notice?

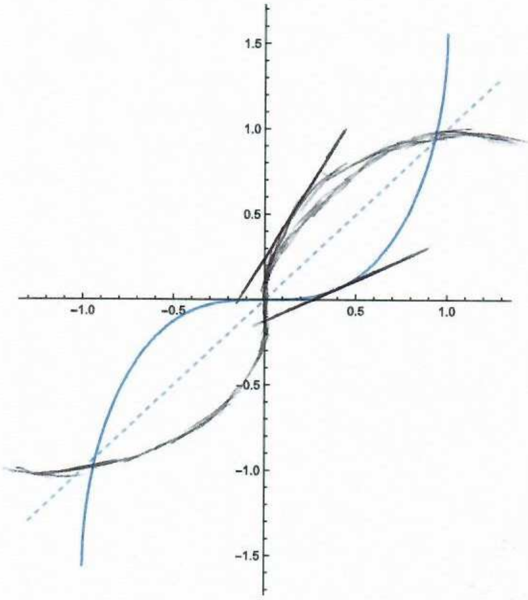
I notice that the tangent line of the inverse function at $y=0.5$ is the reflection of the tangent line of $\arcsin(x^3)$ at $x=0.5$ around $y=x$.

-2 Equation?

$y - 0.5 = \frac{1}{0.7559}(x - 0.1253)$

18

Problem 2: (20 pts) The graph of an invertible function, $h(x) = \arcsin(x^3)$, is given below:



$$\frac{1}{\sqrt{1-g^2}} \cdot 3x^2$$

$$\frac{1}{\sqrt{1-(x^3)^2}} \cdot 3x^2$$

$$f'_x = \frac{3x^2}{\sqrt{1-x^6}}$$

a. (4pts) **Carefully** draw the inverse function h^{-1} into the graph. ✓

b. (6pts) Find the derivative of $h(x)$ at $(0.5, 0.1253)$

$$\frac{3(.5)^2}{\sqrt{1-(.5)^6}} = \frac{3 \cdot \frac{1}{4}}{\sqrt{1-.5^6}} = \frac{3}{4\sqrt{1-.5^6}} = \frac{3}{4\sqrt{\frac{63}{64}}} = \frac{3}{4 \cdot \frac{\sqrt{63}}{8}} = 2$$

$$0.756 \approx \frac{2\sqrt{7}}{7} = \frac{2}{\sqrt{7}} = \frac{6}{3\sqrt{7}} = \frac{6}{\sqrt{63}} = \frac{3}{\frac{\sqrt{63}}{8}}$$

✓ Nice simplification

c. (5 pts) Draw the tangent line at $x = 0.5$ onto the graph of h, and find its equation.

$$(y - 0.1253) = \frac{2\sqrt{7}}{7}(x - 0.5)$$ ✓

−1.5

This would go through the origin....

d. (5 pts) Draw the tangent line to the graph of the inverse function at $y = 0.5$, and write its equation (you don't need to do any additional calculations!). What do you notice?

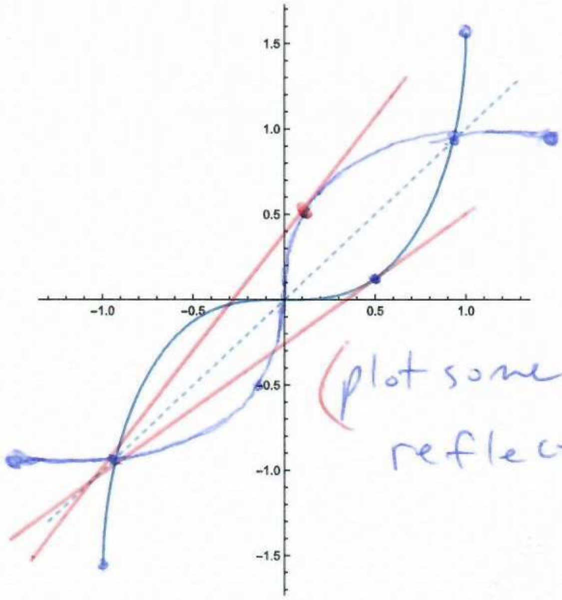
$$(y - 0.5) = \frac{7}{2\sqrt{7}}(x - 0.1253)$$ ✓

−1.5

✓

They are reflected about the x, y axis

Problem 2: (20 pts) The graph of an invertible function, $h(x) = \arcsin(x^3)$, is given below:



(plot some points & reflect them)

$h(x) = y$

$\sin(h(x)) = x^3$

$\sqrt[3]{\sin(h(x))} = x$

$\uparrow$

y

$\underbrace{h^{-1}(x) = (\sin(x))^{1/3}}$

(You did not need to do this!)

a. (4pts) **Carefully** draw the inverse function h^{-1} into the graph.

b. (6pts) Find the derivative of $h(x)$ at $(0.5, 0.1253)$

$$h'(x) = \frac{1}{\sqrt{1-(x^3)^2}} \cdot 3x^2 = \frac{3x^2}{\sqrt{1-x^6}}$$

$$h'(0.5) = 0.755929$$

c. (5 pts) Draw the tangent line at $x = 0.5$ onto the graph of h, and find its equation.

$$L_h(x) = 0.1253 + 0.755929\,(x - 0.5)$$

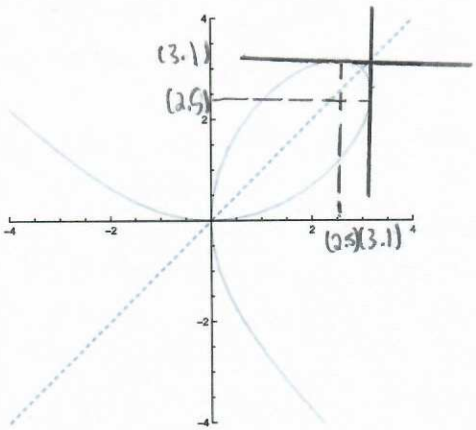
d. (5 pts) Draw the tangent line to the graph of the inverse function at $y = 0.5$, and write its equation (you don't need to do any additional calculations!). What do you notice?

$$L_{h^{-1}}(x) = 0.5 + \frac{1}{0.755929}(x - 0.1253)$$

The tangent lines are also reflected in the mirror of $y = x$.

Problem 3: (20 pts) In class we considered the folium of Descartes, whose equation and graph are

$$x^3 + y^3 = 6xy \tag{1}$$



$$(x^3+y^3)' = (6xy)'$$

$$3x^2 + 3y(x)^2 \cdot y'(x) = 6\,(y(x) + xy'(x))$$

$$3y^2 + y' - 6xy' = 6y - 3x^2$$

$$(3y^2 - 6x)\,y' = 6y - 3x^2$$

$$y' = \frac{6y - 3x^2}{3y^2 - 6x}$$

$$y' = \frac{2y - x^2}{y^2 - 2x}$$

a. (10 pts) Use the space above to compute y' – then write your answer here:

$$\boxed{y' = \frac{2y - x^2}{y^2 - 2x}}$$

✓

b. Find all points (x, y) where the slope of the tangent line is either 0 or ∞. One place where this happens is at the origin: what's going on there? (1 pt)

✓ At (0,0) the "function" hits it's self causing it to be both 0 and ∞.

$y'| = \infty$ (3.1, 2.5) ✓

$y'| = 0$ (2.5, 3.1)

There are two other points where we have 0 or ∞ slopes; but if you find **one**, then you've found the **other** (by reflection!). Here's how you do it:

i. (5 pts) Find the value of y as a function of x that produces zero slope (use your derivative equation!), and then plug that back into the original equation (1).

ii. (2 pts) Then solve this new equation for x. You've found a pair (x, y).

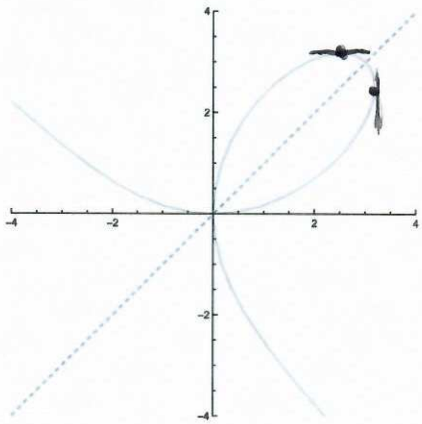
-4.5

good estimates – now find the results analytically

iii. (2 pts) Now use symmetry to find the other point.

Problem 3: (20 pts) In class we considered the folium of Descartes, whose equation and graph are

$$x^3 + y^3 = 6xy \tag{1}$$



$$3x^2 + 3y^2 y' = 6(y + xy')$$

$$(3y^2 - 6x)\, y' = 6y - 3x^2$$

$$y' = -\frac{6y - 3x^2}{6x - 3y^2} = -\frac{3(2y - x^2)}{3(2x - y^2)}$$

Simplify!

a. (10 pts) Use the space above to compute y' – then write your answer here:

$$y' = -\frac{2y - x^2}{2x - y^2}$$

b. Find all points (x, y) where the slope of the tangent line is either 0 or ∞. One place where this happens is at the origin: what's going on there? (1 pt)

The slope is both infinite + zero! Twice the curve passes through.

There are two other points where we have 0 or ∞ slopes; but if you find **one**, then you've found the **other** (by reflection!). Here's how you do it:

i. (5 pts) Find the value of y as a function of x that produces zero slope (use your derivative equation!), and then plug that back into the original equation (1).

$0 = 2y - x^2 \quad y = \frac{1}{2}x^2$

$x^3 + \left(\frac{x^2}{2}\right)^3 = 3x^3$

ii. (2 pts) Then solve this new equation for x. You've found a pair (x, y).

$1 + \frac{x^3}{8} = 3$

$x^3 = 16$

$x = 16^{1/3} \simeq 2.52$

$(16^{1/3}, \frac{1}{2} 16^{2/3})$

iii. (2 pts) Now use symmetry to find the other point.

$(3.17480, 2.51984) \simeq (\frac{1}{2}16^{2/3}, 16^{1/3})$ $(2.51984, 3.17480)$

Problem 4: (10 pts) Find the equation of the tangent line to the graph of

$$f(x) = \frac{3\tan(x)}{1+\sin(x)}$$

when $x = \frac{\pi}{4}$. Write the equation in point-slope form.

$$f(x) = \frac{3\tan(x)}{1+\sin(x)} = \frac{a}{b}$$

$$a' = (3\tan(x))' = 3\sec^2(x)$$

$$b' = (1+\sin(x))' = \cos(x)$$

quotient rule

$$\frac{a'b - ab'}{v^2}$$

$$f'(x) = \frac{3\sec^2(x)(1+\sin(x)) - (3\tan(x))(\cos(x))}{(1+\sin(x))^2}$$

$$f'\left(\frac{\pi}{4}\right) = \frac{3\sec^2\left(\frac{\pi}{4}\right)\left(1+\sin\left(\frac{\pi}{4}\right)\right) - \left(3\tan\frac{\pi}{4}\right)\left(\cos\frac{\pi}{4}\right)}{\left(1+\sin\left(\frac{\pi}{4}\right)\right)^2}$$

$$= \frac{3\left[(\sqrt{2})^2 + \frac{\sqrt{2}}{2} - 1 \cdot \frac{\sqrt{2}}{2}\right]}{\left(1+\frac{\sqrt{2}}{2}\right)^2}$$

← If you factor 3 out, you need parentheses!

$$= \frac{6}{\left(1+\frac{\sqrt{2}}{2}\right)^2} = \frac{6}{\frac{3}{2}+\sqrt{2}}$$

$$= \frac{12}{3+2\sqrt{2}}$$

$$f\left(\frac{\pi}{4}\right) = \frac{3\tan\left(\frac{\pi}{4}\right)}{1+\sin\left(\frac{\pi}{4}\right)}$$

$$= \frac{3(1)}{1+\frac{\sqrt{2}}{2}}$$

$$= \frac{3 \cdot 2}{2+\sqrt{2}}$$

$$= \frac{6}{2+\sqrt{2}}$$

missing something

Point-Slope form

$$y - \frac{6}{2+\sqrt{2}} = \frac{6}{\frac{3}{2}+\sqrt{2}}\left(x - \frac{\pi}{4}\right)$$

-1

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Problem 4: (10 pts) Find the equation of the tangent line to the graph of

$$f(x) = \frac{3\tan(x)}{1+\sin(x)}$$

when $x = \frac{\pi}{4}$. Write the equation in point-slope form.

$\sin(\pi/4) = \cos(\pi/4)$
$= \sqrt{2}/2$

$$f'(x) = \frac{[1+\sin(x)]\, 3\sec^2 x \;-\; 3\tan\pi/4\,\cos(x)}{(1+\sin(x))^2}$$

$$f'(\pi/4) = \frac{(1+\sqrt{2}/2)\,6 \;-\; 3\sqrt{2}/2}{(1+\sqrt{2}/2)^2}$$

$$m = \frac{6+3\sqrt{2}/2}{(1+\sqrt{2}/2)^2} \approx 2.7868$$

$$f(\pi/2) \approx 1.75736 = y_0$$

$$y - 1.75736 = 2.7868\,(x - \pi/4)$$