Section Summary: 2.1

a. Definitions

- tangent - from the latin, *tangens* (touching). The tangent line is a line that just touches a curve at a point.
- secant line - a line passing through two points along the arc of a curve.
- limit notation - in this section, the notation

$$\lim_{Q \to P} m_{PQ} = m$$

means that point Q of the graph will tend toward P of the graph, and as it does the slope m_{PQ} will tend toward a value which represents the slope of the tangent line to the curve at point P.

tangent line - the line through a point on a graph, $P(a, f(a))$, with slope

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

provided the limit exists. [What could possibly go wrong? Well, the function might not be defined at a - i.e., $f(a)$ is undefined; or the function might not be continuous, so that the numerator doesn't tend to 0 (whereas the denominator always does).]

The second definition is obtained from the first via a *change of variables*: if we replace the independent variable x by the independent variable h, linking them by a shift - $x = h + a$ - then we get the second form of the definition. **I consider this definition the most important concept in calculus**!

- **slope of the curve** at a - the slope m of the tangent line at point $P(a, f(a))$. Slope is a concept we've associated with straight lines, not with curves. This gives us a way of associating the idea with curves.

- **position function** - $s = f(t)$ - the location of an object as a function of time. Sometimes we represent it as $s(t)$. Of course, what letter we associate with it (s versus f versus any other letter) is irrelevant: it's just a symbol we prefer to use....

- **instantaneous velocity**

$$v(a) = \lim_{h \to 0} \frac{s(a+h) - s(a)}{h}$$

(the slope of the tangent line to the graph of the position function at $P(a, s(a))$). We can think of this as our first (and one of our most important) examples of the use of this limit.

- **increment** of x - the change in x: $\Delta x = x_2 - x_1$ (usually considered to be a small quantity - that is, the two points x_2 and x_1 are considered to be close together).

- **average rate of change** - $\frac{\Delta y}{\Delta x}$ - the slope of the secant line connecting two points on the graph, $P(x_1, y_1 = f(x_1))$ and $P(x_2, y_2 = f(x_2)))$. This is often used as an approximation to the instantaneous rate of change.

- **instantaneous rate of change** - $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$.

- **derivative of a function** f **at** a - denoted by $f'(a)$,

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

  provided the limit exists. That's right - this is exactly the same as the slope of the tangent line. So the derivative of a function f at a point a is the slope of the tangent line to the curve at $P(a, f(a))$.
  Alternatively, it can be considered the instantaneous rate of change of $y = f(x)$ with respect to x when $x = a$.

b. Properties/Tricks/Hints/Etc.

- Here are some nice examples of rates of change from the areas a student is likely to consider for employment someday:
  - From physics: power - rate of change of work with respect to time
  - From chemistry: rate of reaction - rate of change of chemical concentration with respect to time
  - From business: marginal cost - rate of change of cost of production with respect to some variable x.
- If the tangent line exists, then as we zoom in on the graph at the point $P(a, f(a))$, we discover that the graph of the function and the graph of the tangent line seem to become indistinguishable.

Sometimes the derivative is merely estimated from data, using average rates of change, or by a visual approximation based on a graph.

c. Summary

Here we catch a glimpse of the importance of limits: they are used to define instantaneous velecity and other important rates of change, which prove important in the sciences, engineering, and the social sciences (e.g. power in physics, rates of reaction in chemistry, and marginal costs in business). Furthermore, we should note that the definition of these slopes involves a limit of a quotient whose denominator has a limit of 0: hence, a trick (e.g. simplifying the numerator, or rationalizing) will be necessary in order to evaluate the limit.

The big picture is that the derivative, certainly one of the fundamental concepts of calculus, is actually just the same as the slope of a tangent line to a curve. It can also be considered an instantaneous rate of change.