# MAT129 Test 1 (Spring 2018): Functions, Limits, and Derivatives

**Name:**

**Directions**: All problems are equally weighted. Show your work! Answers without justification will likely result in few points. Your written work also allows me the option of giving you partial credit in the event of an incorrect final answer (but good reasoning). Indicate clearly your answer to each problem (e.g., put a box around it). **Note**: you **must** skip **one** of the eight problems (except for Problem 2). Write "skip" across it, so I'll easily know which one you are skipping. **Good luck!**

**Problem 1** (10 pts) Let $f(x) = \dfrac{x^3}{x^2 - 1}$. **Justify your answers to the following**:

a. How would you describe the function $f$: is it in a particular class that you recognize? What classes of functions make it up?

b. What is its domain of definition? Where is it defined?

c. Where is it continuous? If it is discontinuous at a point, describe the type of discontinuity.

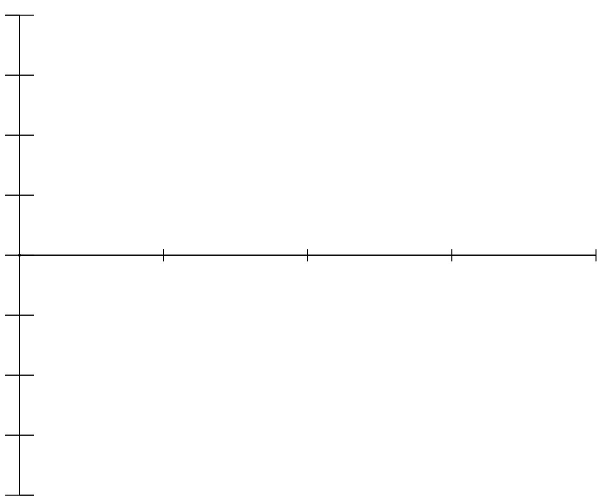
d. Where is it differentiable?

**Problem 2** (10 points – you may not skip this one!) Consider the function $f(x) = \sqrt{x} - 2$.

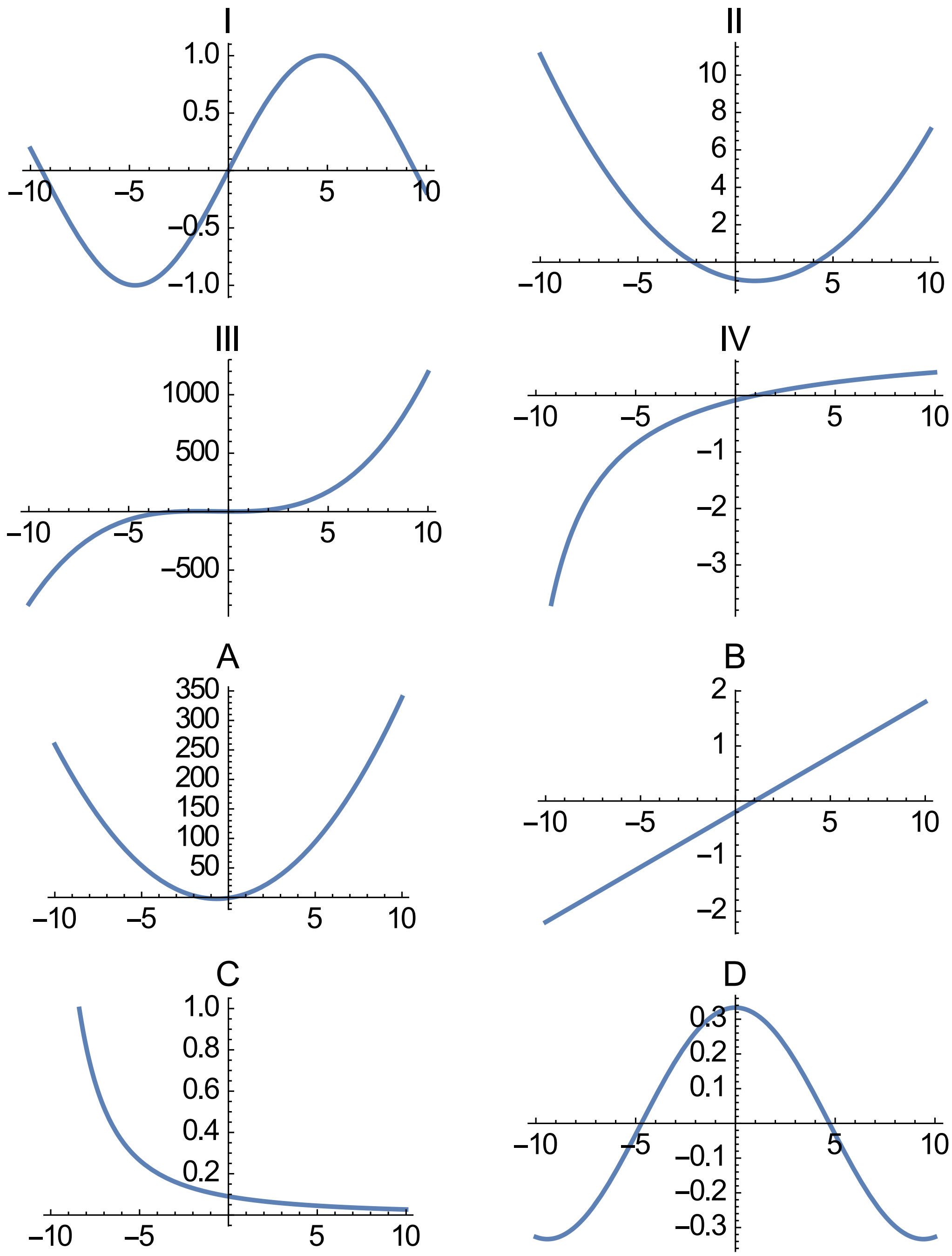
a. (4 pts) Use the limit definition of the derivative to compute $f'(x)$.

b. (3 pts) Use this derivative to write the equation of the tangent line to the function at $(1, -1)$.

c. (3 pts) Graph both $f$ and its tangent line below (label the $y$-axis from -2 to 2):



**Problem 3** (10 points) Match the derivatives to the functions: the functions are on top (labelled I-IV) and their derivatives are below (labelled A-D). Give as many reasons as you can. Feel free to "decorate" the graphs as needed to explain.



**Problem 4** (10 points) Let $f(x) = \dfrac{x^2 - 1}{x + 2}$.

a. (4 pts) Find $\lim_{x \to 2} f(x)$, **given only** the constant limit law $\left(\lim_{x \to c} a = a\right)$, the identity law $\left(\lim_{x \to c} x = c\right)$, and the sum, difference, product, and quotient limit laws. Cite the appropriate limit laws as you go.

b. (2 pts) Explain why you can do this limit by substitution.

c. (4 pts) Describe at least **four** qualitatively different ways in which a limit (in general) may fail to exist. You may illustrate graphically and/or by examples.

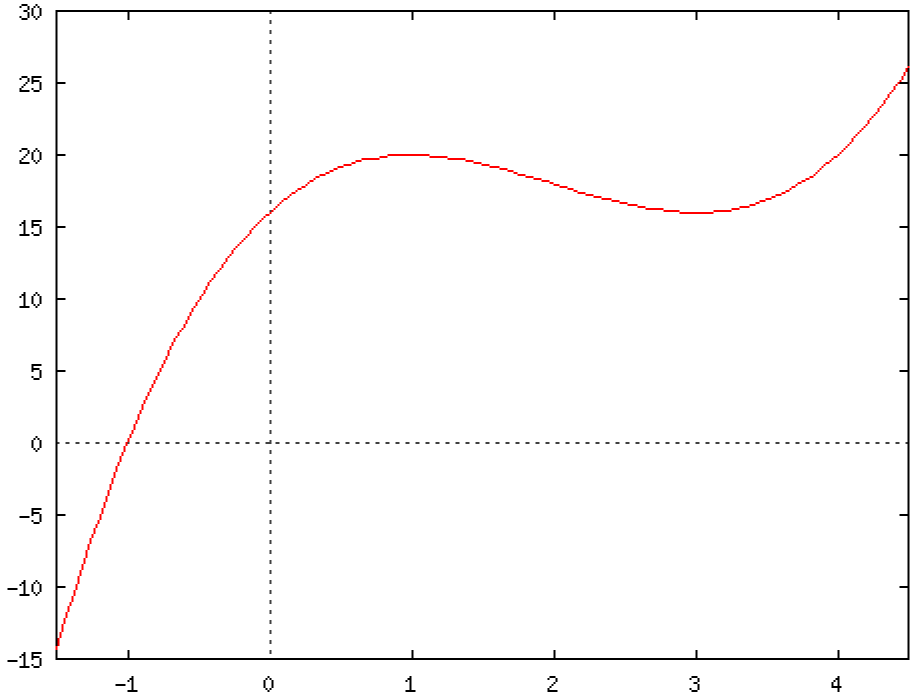
**Problem 5** (10 points) The function $f$ is defined piecewise by three different quadratic functions stitched together – at least they're <u>supposed</u> to be stitched together – into a continuous function.

$$f(x) = \begin{cases} x^2 - 3 & x < 1 \\ ax^2 - bx - 1 & 1 \leq x < 2 \\ (x-1)^2 + 2 & x \geq 2 \end{cases}$$

a. (6 pts) Find values of $a$ and $b$ so that $f$ is indeed a continuous function.

b. (4 pts) Use the intermediate value theorem to **prove** that this function has a root on the interval $(1, 2)$ – that is, prove that there exists $c \in (1, 2)$ such that $f(c) = 0$.

**Problem 6** (10 points) Consider the following graph of a function $f$:




a. (6 pts) Draw in tangent lines to the curve at the points $x = -1, 0, 1, 2, 3, 4$, and give your estimates for their slopes in the table below:

Table 1: Fill in your slopes here (do your calculations for the slopes to the right of the table):

| $x$ | $m$ |
|---|---|
| -1 | |
| 0 | |
| 1 | |
| 2 | |
| 3 | |
| 4 | |

b. (4 pts) Use estimates of the slopes of the tangent lines obtained above to construct a graph of the derivative function $f'(x)$. Draw it on the graph of $f$, using the same scale.

When you're done, explain why your graph makes sense.

**Problem 7** (10 points) If a ball is thrown into the air from a height of 2 meters with a velocity of 30 m/s, its height (in meters) after $t$ seconds is given by

$$s(t) = 2 + 30t - \frac{9.81}{2}t^2$$

a. (6 pts) Find its velocity when $t = 2$ seconds, using the limit definition of the derivative.

b. (4 pts) Write the equation of the tangent line to the graph of $s$ at $t = 2$ seconds.

**Problem 8** (10 points) Short answer:

a. Write a technically correct definition of a function.

b. What is the most important definition in calculus, according to Prof. Long?

c. True or False (and explain!): Differentiability implies continuity, and continuity implies differentiability.

d. Which classes of functions can you name that are continuous and differentiable on their domains?