Section Summary: 1.6

a. Definitions

- **Greatest integer function** (p. 68): $[[x]]$ is the largest integer that is less than or equal to x. This is one of the most important step functions (aka "the floor function". There is also a "ceiling function" – how do you think it's defined?).

b. Theorems

- Suppose that $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. Then

$$\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$$
$$\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$$
$$\lim_{x\to a}[cf(x)] = c\lim_{x\to a} f(x)$$
$$\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x)\lim_{x\to a} g(x)$$
$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} \text{ if } \lim_{x\to a} g(x) \neq 0$$

  These rules all follow your intuition, which is a wonderful thing. For the first one, for example, we could say in words that

  "The limit of the sum is the sum of the limits."

  To use a technical mathematical term, you could even say that the mathematical notions "commute".

  Then there are other very reasonable rules. Like, in many cases, you can "pass a limit inside" a function:

- $$\lim_{x\to a}[f(x)]^n = [\lim_{x\to a} f(x)]^n$$

  where n is a positive integer (the limit of a power is the power of the limit). "The limit of the power is the power of the limit." Again, these notions commute.

- Two special limits:

$$\lim_{x\to a} c = c \text{ and } \lim_{x\to a} x = a$$

  They're obvious graphically, of course. The graphs of $f(x) = c$ and $f(x) = x$ are just clean, beautiful, straight lines, and in each case we see that

$$\lim_{x\to a} f(x) = f(a)$$

  This is a very important happenstance, which is called "continuity" (to be studied in detail in section 1.8).

- Using the second special limit and the preceding properties,

$$\lim_{x \to a} x^n = a^n$$

where n is a positive integer.

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$$\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$$

where n is a positive integer (and $a > 0$ if n is even).

-

$$\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$$

where n is a positive integer. Again, we can "pass the limit inside".

- If f is a polynomial or a rational function and a is in the domain of f, then

$$\lim_{x \to a} f(x) = f(a)$$

For these functions, computing limits is easy! Furthermore, it says that each is continuous on its domain (again, more to come in section 1.8).

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$$\lim_{x \to a} f(x) = L \text{ if and only if } \lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x)$$

- If $f(x) \leq g(x)$ when x is near a (except possibly at a) and the limits of f and g both exist as x approaches a, then

$$\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x)$$

- **Squeeze theorem** or **Pinching theorem**: if $f(x) \leq g(x) \leq h(x)$ when x is near a (except possibly at a) and

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

then

$$\lim_{x \to a} g(x) = L.$$

The function g is stuck between f and h, and as the functions f and h tend to the same value, g has nowhere to go but the same place!

c. Properties/Tricks/Hints/Etc.

- All the properties noted above apply to one-sided limits as well.
- Sometimes simplifying an expression (e.g. Example 5, p. 66) or rationalizing an expression (e.g. Example 6, p. 66) makes computation of limits easier.
- There's a nice historical note on Isaac Newton on p. 64. Born on Christmas day, 1642.... in the year that Galileo Galilei died.

d. Summary

Many properties of limits are very common sense: sums, differences, products, quotients, powers, roots, etc. are computed simply. It is especially easy to compute limits as $x \to a$ for important classes of functions like polynomials and rational functions: simply evaluate the function at a, $f(a)$! The most interesting theorem in this section is probably the pinching theorem, and the idea of squeezing a function between two others and deducing properties of the squeezed function from their behavior is very interesting.

Problems we might do together:

pp. 69-71, #2, 7, 10, 25, 33, 47, 54