Section Summary: 1.8

Continuous functions have graphs with a strong connectivity. It is frequently said that one can graph them without lifting the pencil from the paper, and that's an important way of thinking about them.

a. Definitions

- A function f is **continuous** at $x = a$ if

$$\lim_{x \to a} f(x) = f(a)$$

  The intuitive idea is that in a small neighborhood of $x = a$, the function f takes on values close to $f(a)$. "Small" is a relative term, however. The point is that, if we get close enough to a, we will find that the function values $f(x)$ are as close as we'd like to $f(a)$.
  As is the case for limits in general, a function can be continuous from the left only,

$$\lim_{x \to a^-} f(x) = f(a)$$

  or from the right only:

$$\lim_{x \to a^+} f(x) = f(a)$$

- We will say that f is **continuous on interval** I if f is continuous at every point on the interval I. An intuitive way of saying this is that the graph of f can be drawn on the interval I without lifting one's pencil from the paper.
- Function f is **discontinuous** at a if it is not continuous at a (so it's a binary world – f is either continuous at a point, or discontinuous there). Types of discontinuities:
  - removable discontinuity - a hole exists in the graph because either $f(a)$ doesn't exist, or is not the same as the existing limit

$$\lim_{x \to a} f(x)$$

    (which must exist for the removable discontinuity).
  - infinite discontinuity - the function has a vertical asymptote at a
  - jump discontinuity -

$$\lim_{x \to a} f(x)$$

    does not exist, because

$$\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$$

    There is a "jump" in the graph.

b. Theorems

- If f and g are continuous at a, then f+g, f-g fg, cf (for $c \in \Re$), and $\frac{f}{g}$ (if $g(a) \neq 0$) are all continuous at a.

  Here again, all the properties we'd like continuous functions to have are indeed satisfied: the sum of continuous functions is continuous, etc.

- Polynomials, rational functions, root functions, and trigonometric functions are all continuous on their domains.

  This is one of the most important theorems in this section: it means that limits are easy to evaluate for these classes of functions – just evaluate the function!

  One caveat, however: root functions may be only continuous from one side at zero: e.g.

  $$\lim_{x \to 0} \sqrt{x}$$

  does not exist (because x values to the left of 0 result in complex numbers for $\sqrt{x}$; but

  $$\lim_{x \to 0^+} \sqrt{x}$$

  exists, and

  $$\lim_{x \to 0^+} \sqrt{x} = \sqrt{0}$$

- If f is continuous at b, and $\lim_{x \to a} g(x) = b$, then

  $$\lim_{x \to a} f(g(x)) = f(\lim_{x \to a} g(x)) = f(b)$$

  Note that this is not the same as $f(g(a))$, which may not exist (because $g(a)$ may not exist, even though its limit does). Hence the composition is not necessarily continuous at a.

- If g is continuous at a and f is continuous at $g(a)$, then the composition $f \circ g$ given by

  $$(f \circ g)(x) = f(g(x))$$

  is continuous at a, and

  $$\lim_{x \to a} (f \circ g)(x) = f(g(a))$$

  "A composition of continuous functions is continuous."

- Intermediate Value Theorem - Suppose that f is continuous on $[a, b]$ and let N be any number between $f(a)$ and $f(b)$. Then there is a $c \in (a, b)$ such that $f(c) = N$.

  This theorem effectively says that in crossing a street you have to cross every line between the curbs and parallel to the curbs: you can't get from one side to the other without going through the middle.

c. **Properties/Tricks/Hints/Etc.**

- The graph of a function continuous on an interval can be drawn on that interval without lifting the pen from the paper.
- If f is continuous at a, then
  - $f(a)$ is defined
  - $\lim_{x \to b} f(x)$ exists, and
  - $\lim_{x \to b} f(x) = f(a)$.

d. **Summary**

Continuity is defined and discussed in detail, along with varieties of discontinuities (jump, infinite, removable). Once again the theorems suggest that properties we'd like continuity to possess are true: e.g., the sum of two continuous functions is continuous; the composition of continuous functions is continuous. Entire classes of functions are continuous on their domains (polynomials, rational functions, trigonometric functions, root functions), so limits for these are easy to compute.

Problems for in-class: pp. 90-93, #3, 10, 12, 19, 23, 36, 46