Absolute Convergence, and Root and Ratio Tests

MAT 229, Spring 2025

Supporting materials

If you wish to get a different perspective on the notes below, try either of the following textbook sections.

- Strang's *Calculus*
  Vol 2, Section 5.5: Alternating Series (Absolute convergence)
  Vol 2, Section 5.6: Ratio and root tests
- Stewart's *Calculus*
  Section 11.6: Absolute convergence and the ratio and root tests
- Boelkins/Austin/Schlicker's Active Calculus
  Section 8.3: Series of real numbers

Review

Question

The alternating series test guarantees $\sum_{k=1}^{\infty} (-1)^{k+1} a_k$ converges if what is true about a_k? (Three things....)

Questions

- Does the p-series $\sum_{k=1}^{\infty} \frac{1}{k}$ converge or diverge? Why or why not?
- Does the alternating p-series $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k}$ converge or diverge? Why or why not?

Questions

- Does the alternating series $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k}{e^k}$ converge or diverge? Why or why not?
- Does the alternating series $\sum_{k=1}^{\infty} (-1)^{k+1} \cos\left(\frac{1}{k}\right)$ converge or diverge? Why or why not?
- Does the series $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{\sin(k)}{k^2}$ converge or diverge? Why or why not?

Alternating series error estimate

Questions

- What is the error in approximating the alternating series $\sum_{k=1}^{\infty} (-1)^{k+1} a_k$ with its n^{th} partial sum?
- What is an easy estimate on this error?

Questions

- What partial sums approximate $\sum_{k=1}^{\infty} (-1)^k \frac{1}{k}$ with error less than 0.001?
- Approximate $\sum_{k=1}^{\infty} (-1)^k \frac{1}{k^3}$ with error less than 0.001.

Absolute convergence test

What if a series has some positive and some negative terms and neither the divergence test nor the alternating series test are applicable?

Absolute convergence test

If $\sum_{k}^{\infty} \left| b_k \right|$ converges then $\sum_{k}^{\infty} b_k$ converges, and we say that it converges **absolutely**. If $\sum_{k}^{\infty} \left| b_k \right|$ diverges but $\sum_{k}^{\infty} b_k$ converges, then $\sum_{k}^{\infty} b_k$ converges, but it does so only **conditionally**.

A given series either converges absolutely, converges conditionally, or it diverges.

Question

We have discussed a series that meets the definition of conditional convergence. What is it?

Question

Determine whether the series converges absolutely, converges conditionally, or diverges.

- $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k}{k^2+1}$
- $\sum_{k=0}^{\infty} (-1)^k \frac{e^k}{k+1}$
- $\sum_{k=1}^{\infty} (-1)^{k-1} \frac{3k^2-1}{5k^5+k^3-2k}$

Questions

Consider the series $1 + \frac{1}{4} - \frac{1}{9} - \frac{1}{16} + \frac{1}{25} + \frac{1}{36} - \frac{1}{49} - \frac{1}{64} + \ldots$

- Does the divergence test imply anything about this sequence?
- Is the alternating series test applicable?
- Is the absolute convergence test applicable?

- Does this series converge or diverge?

Questions

Consider the series $\sum_{k=1}^{\infty} \frac{\cos(k)}{k^3}$.

- Begin by applying the absolute convergence test to look at $\sum_{k=1}^{\infty} \frac{|\cos(k)|}{k^3}$. What are the first 5 partial sums of this series?
- Comparing this series to one you are familiar with, can you find a bound for the sequence of partial sums?
- Does $\sum_{k=1}^{\infty} \frac{|\cos(k)|}{k^3}$ converge?
- Does $\sum_{k=1}^{\infty} \frac{\cos(k)}{k^3}$ converge?
- How can we estimate the error in approximating $\sum_{k=1}^{\infty} \frac{\cos(k)}{k^3}$ with $\sum_{k=1}^{10} \frac{\cos(k)}{k^3}$?

Ratio and root tests

Two tests that will be the workhorses for our future analyses are based on comparisons to geometric series. They incorporate the absolute convergence test.

Questions

- What is the form of a geometric series?
- Under what circumstances does a geometric series converge or diverge?

Questions

- Let c_k be the k^{th} term of a geometric series. What is $\frac{c_{k+1}}{c_k}$? What is $\lim_{k\to\infty} \frac{c_{k+1}}{c_k}$? What about $\lim_{k\to\infty} \frac{c_{k+1}}{c_k}$ Implies the convergence of $\sum_k^{\infty} c_k$?
- Let c_k be the k^{th} term of a geometric series. What is $\sqrt[k]{c_k}$? What is $\lim_{k\to\infty} \sqrt[k]{c_k}$? What about $\lim_{k\to\infty} \sqrt[k]{c_k}$ Implies the convergence of $\sum_k^{\infty} c_k$?

Ratio test

Given any series $\sum_k^{\infty} b_k$, evaluate the limit of the ratio of successive terms, ignoring any signs, $\lim_{k\to\infty} \left| \frac{b_{k+1}}{b_k} \right|$.

- If $\lim_{k\to\infty} \left| \frac{b_{k+1}}{b_k} \right| = L$ where $L < 1$, then $\sum_k^{\infty} b_k$ converges.
- If $\lim_{k\to\infty} \left| \frac{b_{k+1}}{b_k} \right| = L$ where $L > 1$, then $\sum_k^{\infty} b_k$ diverges.
- If $\lim_{k\to\infty} \left| \frac{b_{k+1}}{b_k} \right| = 1$, then you must use another convergence test. The series doesn't compare to a geometric series.

Questions

Apply the ratio test to determine the convergence of the following series.

- $\sum_{k=1}^{\infty} \frac{k}{4^k}$
- $\sum_{k=1}^{\infty} \frac{(-3)^k}{k!}$

Root test

Given any series $\sum_{k}^{\infty} b_k$, evaluate the limit of the k^{th} root, ignoring any signs, of the k^{th} term, $\lim_{k \to \infty} \sqrt[k]{|\, b_k \,|}$.

- If $\lim_{k \to \infty} \sqrt[k]{|\, b_k \,|} = L$ where $L < 1$, then $\sum_{k}^{\infty} b_k$ converges.
- If $\lim_{k \to \infty} \sqrt[k]{|\, b_k \,|} = L$ where $L > 1$, then $\sum_{k}^{\infty} b_k$ diverges.
- If $\lim_{k \to \infty} \sqrt[k]{|\, b_k \,|} = L$ where $L = 1$, then you must use another convergence test. The series doesn't compare to a geometric series.

Questions

Apply the root test to determine the convergence of the following series.

- $\sum_{k=1}^{\infty} \frac{k}{4^k}$
- $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{(k^2+k+3)^k}{100^k}$