Alternating Series

MAT 229, Spring 2025

Supporting materials

If you wish to get a different perspective on the notes below, try either of the following textbook sections.

- Strang's Calculus:
  Vol 2, Section 5.5: Alternating Series
- Stewart's *Calculus*
  Section 11.5: Alternating series
- Boelkins/Austin/Schlicker's Active Calculus
  Section 8.4: Alternating series

Review

Series

The infinite sum $\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \ldots$ converges or diverges if its partial sums converge or diverge.

Question

If the series $\sum_{k=1}^{\infty} a_k$ converges, what must be true about the sequence a_k, $k = 1, 2, \ldots$?

Question

What particular series do we use for comparisons?

Question

What tests for convergence and divergence have we discussed?

Alternating series

So far we have looked mostly at series where we sum nonnegative terms. We want to explore what happens if some of the terms are negative.

Alternating Series Test (AST):

The alternating series $\sum_{k=1}^{\infty} (-1)^{k+1} a_k$ converges if

1. $a_k \geq 0$ for $k = 1, 2, 3, \ldots$
2. The sequence $\{a_k\}$ is a decreasing sequence.
3. $\lim_{k\to\infty} a_k = 0$

Questions

- Does the p-series $\sum_{k=1}^{\infty} \frac{1}{k}$ converge or diverge? Why or why not?
- Does the alternating p-series $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k}$ converge or diverge? Why or why not?

Questions

- What are the first four partial sums of the p-series $\sum_{k=1}^{\infty} \frac{1}{k}$?
- What are the first four partial sums the alternating p-series $\sum_{k=1}^{\infty} (-1)^{k} \frac{1}{k}$?

Questions

- Does the alternating series $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k!}$ converge or diverge? Why or why not?
- Does the alternating series $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k-1}{k^2}$ converge or diverge? Why or why not?

Notes

- If the three conditions are met
  - the series is truly alternating,
  - the individual terms without the sign are decreasing, and
  - the limit of the individual terms go to zero,

  then the alternating series converges.
- If the last condition is *not* met, $\lim_{n\to\infty} a_n = 0$, then the series diverges by the divergence test.
- If either of the first two conditions are not met, the alternating series test provides no conclusion: we don't yet know whether the series converges or not.

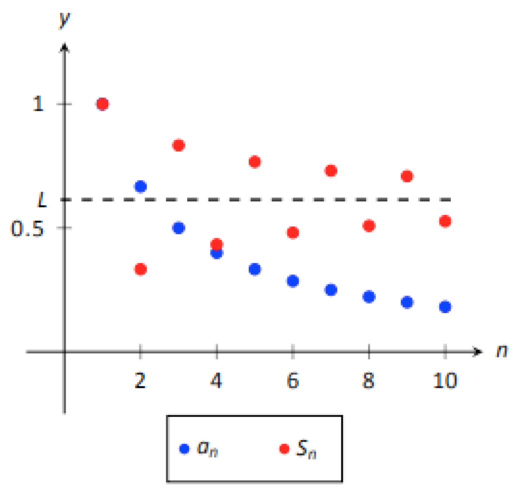
Questions

- Does the series $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k+1}{k}$ converge or diverge? Why or why not?
- Does the series $1 + \frac{1}{2} - \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \frac{1}{6} - \frac{1}{7} - \frac{1}{8} + \ldots$ converge or diverge? Why or why not?
- Does the series $1 - \frac{1}{3} + \frac{1}{2} - \frac{1}{5} + \frac{1}{4} - \frac{1}{7} + \frac{1}{6} - \frac{1}{9} + \ldots$ converge or diverge? Why or why not?

Alternating series error estimate

We approximate series using partial sums. In the case of a converging alternating series, with decreasing terms (per the AST):

$$\sum_{k=1}^{\infty} (-1)^{k+1} a_k \approx \sum_{k=1}^{n} (-1)^{k+1} a_k = a_1 - a_2 + a_3 - a_4 + \ldots + (-1)^{n+1} a_n$$



Questions

- What is the error in approximating the alternating series $S = \sum_{k=1}^{\infty} (-1)^{k+1} a_k$ with its n^{th} partial sum, $S_n = \sum_{k=1}^{n} (-1)^{k+1} a_k$?

Absolute error = $|S - S_n| = |\sum_{k=n+1}^{\infty} (-1)^{k+1} a_k|$

- What is an estimate on this error?

Alternating test error estimate

If the series $\sum_{k=1}^{\infty} (-1)^{k+1} a_k$ meets the requirements of the alternating series test so that it converges, then the error in approximating it with its n^{th} partial sum $\sum_{k=1}^{n} (-1)^{k+1} a_k$ is given very simply:

$$\text{error} \le a_{n+1}$$

Questions

- What is the error in approximating $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k^2+3k+5}$ with $\sum_{k=1}^{10} (-1)^{k+1} \frac{1}{k^2+3k+5}$?
- Which partial sums approximate $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k}$ with error less than 0.001?
- Approximate $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k^3}$ with error less than 0.001.