Improper Integration

MAT 229, Spring 2025

Supporting materials

If you wish to get a different perspective on the notes below, try either of the following textbook sections.

- Strang's Calculus:
  Vol 2, Sec 3.7: Improper Integrals
- Stewart's *Calculus*
  Section 7.8: improper Integrals
- Boelkins/Austin/Schlicker's Active Calculus
  Section 6.5: Improper Integrals

Improper integrals

An improper integral is a definite integral where either one or both of the limits is ∞, or the integrand is not defined for some value(s) of x between the limits of integration. Frequently there is a vertical asymptote causing trouble.

- Proper integral: $\int_1^{10} \frac{1}{x^2}\,dx$
- Improper integral: $\int_1^{\infty} \frac{1}{x^2}\,dx$
- Improper integral: $\int_0^{10} \frac{1}{x^2}\,dx$

We **make sense** out of an improper integral by **turning it into a limit:**

- Improper integral:
  $$\begin{aligned}\int_1^{\infty} \frac{1}{x^2}\,dx &= \lim_{R\to\infty} \int_1^{R} \frac{1}{x^2}\,dx \\ &= \lim_{R\to\infty} \int_1^{R} x^{-2}\,dx \\ &= \lim_{R\to\infty} \left(-x^{-1}\ \Big|_1^R\right) \\ &= \lim_{R\to\infty} \left(-R^{-1} + 1^{-1}\right) \\ &= \lim_{R\to\infty} \left(-\frac{1}{R} + 1\right) = 0 + 1 = 1\end{aligned}$$
  This improper integral converges to 1.
- Improper integral:
  $$\begin{aligned}\int_0^{10} \frac{1}{x^2}\,dx &= \lim_{r\to 0^+} \int_r^{10} \frac{1}{x^2}\,dx \\ &= \lim_{r\to 0^+} \int_r^{10} x^{-2}\,dx \\ &= \lim_{r\to 0^+} \left(-x^{-1}\ \Big|_r^{10}\right) \\ &= \lim_{r\to 0^+} \left(-10^{-1} + r^{-1}\right) \\ &= \lim_{r\to 0^+} \left(-\frac{1}{10} + \frac{1}{r}\right) = -\frac{1}{10} + \infty\end{aligned}$$
  This improper integral diverges to ∞.

Questions

- What makes $\int_0^\infty \frac{1}{1+x^2}\,dx$ an improper integral? Does it converge or not?
- What makes $\int_0^1 \frac{1}{\sqrt{x}}\,dx$ an improper integral? Does it converge or not?
- What makes $\int_0^2 \frac{1}{(x-1)^{2/3}}\,dx$ an improper integral? Does it converge or not?

Questions

1. Let S be the region between the x-axis and $y = \frac{1}{x}$ for $x \geq 1$.

- Is the area of S finite or not?
- Rotate S about the x-axis to create a solid of revolution. Is the volume of this solid finite or not?
- Does this result seem mysterious to you? The area of a cross-section infinite, but the volume of revolution of that area finite?

2. Let's investigate various powers of x:

- Evaluate $\int_1^\infty \frac{1}{x^3}\,dx$, $\int_1^\infty \frac{1}{x}\,dx$, and $\int_1^\infty \frac{1}{x^{1/3}}\,dx$.
- For which values of p does $\int_1^\infty \frac{1}{x^p}\,dx$ converge and for which p does it diverge?
- Evaluate $\int_0^1 \frac{1}{x^3}\,dx$, $\int_0^1 \frac{1}{x}\,dx$, and $\int_0^1 \frac{1}{x^{1/3}}\,dx$.
- For which values of p does $\int_0^1 \frac{1}{x^p}\,dx$ converge and for which p does it diverge?

3. It's not **just** powers of x, of course:

- Evaluate $\int_0^1 ln(x)\,dx$, $\int_0^\infty e^{-x}\,dx$.

The consequence of changing the power on *x*: *n=1* is the borderland....

This first "Manipulate" command shows the consequences of changing the power on *x* in the integral $\int_0^1 \frac{1}{x^p}\,dx$ from n=0.01 to 4:

This second "Manipulate" command shows the consequences of changing the power on *x* in the improper integral $\int_1^\infty \frac{1}{x^p}\,dx$ from n=0.01 to 4:

Things change dramatically at n=1 in both cases. n=1 is a very special value -- **neither** improper integral, $\int_1^\infty \frac{1}{x}\,dx$ or $\int_0^1 \frac{1}{x^p}\,dx$, converges. For all other values of n, one or the other improper integral converges. What else is special about $\frac{1}{x}$? It's the only power without a power as an antiderivative: its antiderivative is *ln(x)*!

And *ln(x)* blows down to -∞ at 0, and blows up to ∞ as x approaches ∞.... That's a very special power, $\frac{1}{x}$!