The Integral Test

MAT 229, Spring 2025

Supporting materials

Review

Question

What are the first four partial sums for $\sum_{k=1}^{\infty} \frac{1}{k^2}$?

Questions

Which of the following converge? To what?

- $\sum_{k=0}^{\infty} 4\left(\frac{3}{2}\right)^k$
- $\sum_{k=0}^{\infty} 5\left(-\frac{2}{3}\right)^k$
- $\frac{7}{4^3} + \frac{7}{4^4} + \frac{7}{4^5} + \frac{7}{4^6} + \ldots = \frac{7}{4^3}\left(1 + \frac{1}{4^1} + \frac{1}{4^2} + \frac{1}{4^3} + \ldots\right)$

Question

How do I know that $\sum_{k=1}^{\infty} \sin(k)$ diverges?

Series Tails

- Finite sums have finite values.

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + \ldots + a_{n-1} + a_n$$

- Infinite sums can be written as the sum of its first few terms and all the other terms

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + \ldots + a_{n-1} + a_n + a_{n+1} + a_{n+2} + a_{n+3} + \ldots$$

$$= \sum_{k=1}^{n} a_k + \sum_{k=n+1}^{\infty} a_k$$

We say $\sum_{k=n+1}^{\infty} a_k$ is a *tail* of the series.

Series convergence

Because the first few terms have a finite sum, the whole series converges if and only if each tail converges.

Integral Test

Questions

Consider the series $\sum_{k=1}^{\infty} \frac{1}{k^2}$. The summands (terms) of this series are $1, \frac{1}{4}, \frac{1}{9}, \ldots$.

- You have computed the first few partial sums for this series. How do the partial sums compare, S_1 with S_2, S_2 with S_3, etc.?
- Is this monotonic or not?

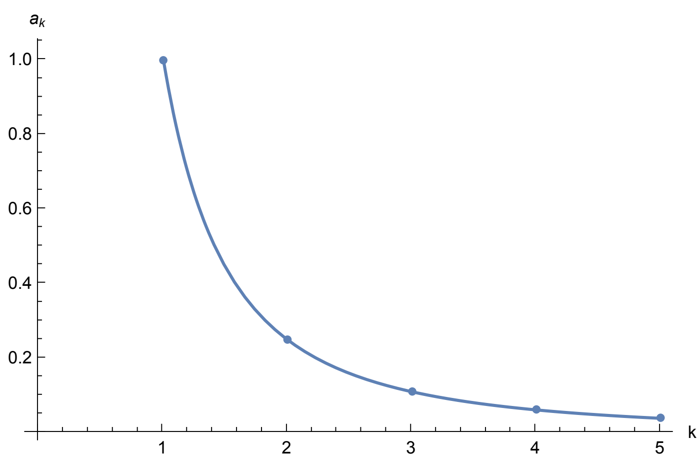
Questions

Suppose the summands $a_1, a_2, a_3, \ldots$ are all positive.

- Are the partial sums $S_n = \sum_{k=1}^{n} a_k$ monotonic?
- If we can find an upper bound for the sequence $S_1, S_2, S_3, \ldots$ what do we know about the convergence of $\sum_{k=1}^{\infty} a_k$?

Questions

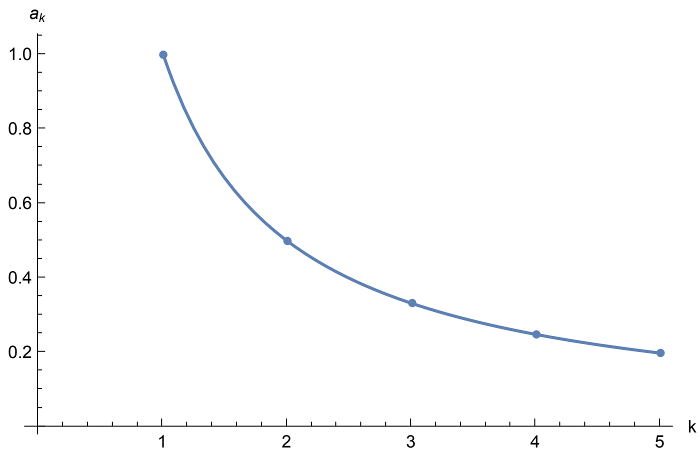
Let $a_k = \frac{1}{k^2}$. The plot of these terms is



- Approximate $\int_1^5 \frac{1}{x^2}\,dx$ using $n = 4$ and the right hand endpoints.
- Is that an underestimate, overestimate, or can't tell?
- Approximate $\int_1^5 \frac{1}{x^2}\,dx$ using $n = 4$ and the left hand endpoints.
- How is that approximation related to $\sum_{k=2}^{5} \frac{1}{k^2}$?
- What is $\int_1^\infty \frac{1}{x^2}\,dx$? What can you conclude about $\sum_{k=1}^{\infty} \frac{1}{k^2}$?

Questions

Let $a_k = \frac{1}{k}$. The plot of these terms is



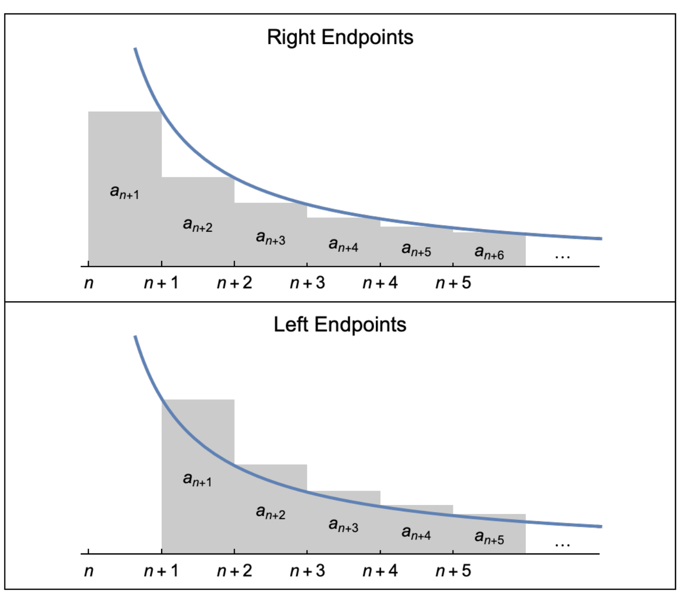
- Approximate $\int_1^5 \frac{1}{x}\,dx$ using $n = 4$ and the *left* hand endpoints.
- Is that an underestimate, overestimate, or can't tell?
- How is that approximation related to $\sum_{k=1}^{4} \frac{1}{k}$?
- What is $\int_1^\infty \frac{1}{x}\,dx$? What can you conclude about $\sum_{k=1}^{\infty} \frac{1}{k}$?

Integral test

Given the infinite series $\sum_{k=n}^{\infty} a_k$, if there is an integrable function $f(x)$ such that

- $f(k) = a_k$ for $k \geq n$,
- $f(x) \geq 0$ for $x \geq n$,
- $f(x)$ is a decreasing function for $x \geq n$.

then the infinite series $\sum_{k=n}^{\infty} a_k$ converges if and only if the improper integral $\int_n^\infty f(x)\,dx$ converges.



$$\sum_{k=n+1}^{\infty} a_k = \text{LRR} > \int_{n+1}^{\infty} f(x)\,dx > \text{RRR} = \sum_{k=n+2}^{\infty} a_k$$

Question

Use the integral test to determine if $\sum_{k=1}^{\infty} e^{-k}$ converges or not.

Question

For which values of p does the infinite series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converge and for which values of p does it diverge?

Error estimate

Once you know a series converges, you can approximate it with a partial sum.

$$\sum_{k=1}^{\infty} a_k \approx \sum_{k=1}^{n} a_k$$

The absolute error in this approximation is

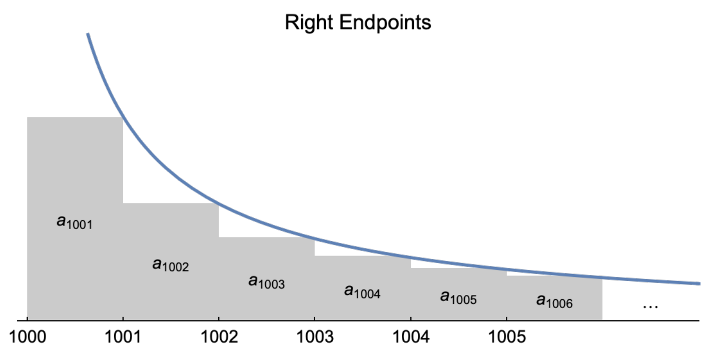
$$\begin{aligned} &\text{error} \\ &= |\text{exact} - \text{approximation}| \\ &= \left|\sum_{k=1}^{\infty} a_k - \sum_{k=1}^{n} a_k\right| \\ &= |(a_1 + a_2 + a_3 + \ldots + a_n + a_{n+1} + a_{n+2} + \ldots) - (a_1 + a_2 + a_3 + \ldots + a_n)| \\ &= |a_{n+1} + a_{n+2} + \ldots| \\ &= \left|\sum_{k=n+1}^{\infty} a_k\right| \end{aligned}$$

The error is just a tail of the series.

Integral test error estimate

If we know a series $\sum_{k=1}^{\infty} a_k$converges due to the integral test with function $f(x)$, then the error in approximating $\sum_{k=1}^{\infty} a_k$ with the partial sum $\sum_{k=1}^{n} a_k$ is

$$\sum_{k=n+1}^{\infty} a_k \leq \int_n^{\infty} f(x)\, dx$$



Questions

Consider the series $\sum_{k=1}^{\infty} \frac{1}{k^3}$. We know this converges by the integral test.

- What is the error in approximating $\sum_{k=1}^{\infty} \frac{1}{k^3}$ with $\sum_{k=1}^{5} \frac{1}{k^3}$?
- How should I choose n to approximate $\sum_{k=1}^{\infty} \frac{1}{k^3}$ with $\sum_{k=1}^{n} \frac{1}{k^3}$ so that the error is no more than 0.0001

Questions

- Does the series $\sum_{k=0}^{\infty} \frac{1}{k^2+1}$ converge or diverge? If it converges, approximate it with a partial sum with error less than 0.001.
- Does the series $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ converge or diverge? If it converges, approximate it with a partial sum with error less than 0.001.