More Improper Integrals

MAT 229, Spring 2025

Supporting materials

If you wish to get a different perspective on the notes below, try either of the following textbook sections.

- Strang's Calculus:
  Vol 2, Sec 3.7: Improper Integrals
- Stewart's *Calculus*
  Section 7.8: improper Integrals
- Boelkins/Austin/Schlicker's Active Calculus
  Section 6.5: Improper Integrals

Improper integrals

An improper integral is a definite integral $\int_a^b f(x)\,dx$ where either

- One or both of a or b are infinite, or
- the integrand $f(x)$ is not defined at a finite number of values of x in interval $[a, b]$.

In either of these cases, turn them into limit problems. If the limit converges, the improper integral has a value. If it diverges, it has no value.

Questions

- $\int_1^\infty \frac{1}{1+x^2}\,dx$ is improper because of the infinite upper limit. Does the integral converge or diverge? If it converges, what is its limit?
- $\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx$ is improper because the integrand is not defined at the upper limit of 1. Does the integral converge or diverge? If it converges, what is its limit?

Examples

We can show that

- $\int_1^\infty \frac{1}{x^p}\,dx \begin{cases} \text{converges} & \text{if } p > 1 \\ \text{diverges} & \text{if } p \le 1 \end{cases}$
- $\int_0^1 \frac{1}{x^p}\,dx \begin{cases} \text{diverges} & \text{if } p \ge 1 \\ \text{converges} & \text{if } p < 1 \end{cases}$
- Notice that $\frac{1}{x}$ (that is, p=1) is the case "on the border": all the other powers converge in one case or the other -- this one is divergent on both. And recall that its antiderivative is *ln(|x|)*, whereas all the other powers have powers as antiderivatives. Coincidence? I think not!:)

Questions

- Does $\int_1^\infty \frac{1}{x^{1.5}}\,dx$ converge or diverge? If it converges, what is its limit?
- Does $\int_0^1 \frac{1}{x^{1.5}}\,dx$ converge or diverge? If it converges, what is its limit?

Comparison Test

Knowing whether an improper integral converges or diverges is useful. If it converges, we can approximate it using numerical integration. However, if we know it diverges, then we know that we cannot approximate it. Comparing a new improper integral to a known improper integral might gives us that information.

Example

The integral $\int_1^\infty \frac{1}{x^3+\sqrt{x}}\,dx$ is improper. Turn it into a limit problem.

$$\int_1^\infty \frac{1}{x^3+\sqrt{x}}\,dx = \lim_{R\to\infty} \int_1^R \frac{1}{x^3+\sqrt{x}}\,dx$$

However, we don't have any handy integration techniques to find an antiderivative of this one. In fact, Mathematica shows that we don't get any **elementary** results for the antiderivative.

On the other hand, the integrand satisfies the inequality

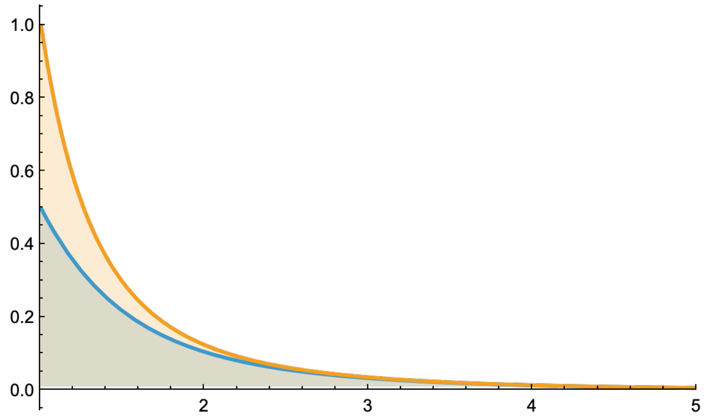
$$\frac{1}{x^3+\sqrt{x}} \le \frac{1}{x^3}$$

for $1 \le x$. (Larger positive denominators mean smaller values.) That means

$$\int_1^\infty \frac{1}{x^3+\sqrt{x}}\,dx \le \int_1^\infty \frac{1}{x^3}\,dx$$

Since we know the larger one converges and represents the area between the x-axis and $y = \frac{1}{x^3}$, the smaller area under $y = \frac{1}{x^3+\sqrt{x}}$ must also converge and has smaller area.

$$\int_1^\infty \frac{1}{x^3+\sqrt{x}}\,dx \le \int_1^\infty \frac{1}{x^3}\,dx = \lim_{R\to\infty} \int_1^R x^{-3}\,dx = \lim_{R\to\infty} \frac{x^{-2}}{-2}\,\Big|_1^R = \lim_{R\to\infty} -\frac{1}{R^2} + \frac{1}{2} = \frac{1}{2}$$



We know that $\int_1^\infty \frac{1}{x^3+\sqrt{x}}\,dx$ converges to some value smaller than $1/2$.

Example

The integral $\int_0^1 \frac{e^x}{x}\,dx$ is improper. Turn it into a limit problem.

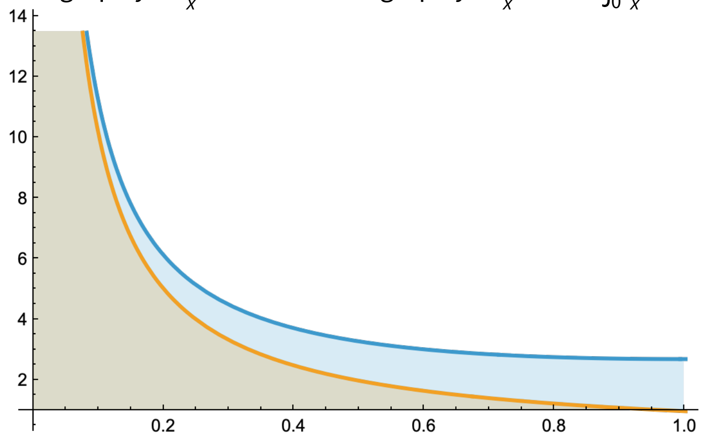
$$\int_0^1 \frac{e^x}{x}\,dx = \lim_{r\to 0} \int_r^1 \frac{e^x}{x}\,dx$$

We don't have any integration techniques to find an antiderivative. As in the last example, Mathematica shows that we don't get any elementary results for the antiderivative.

Here, for $0 \le x \le 1$ the numerator satisfies $1 \le e^x \le e$. In particular, dividing $1 \le e^x$ by x produces

$$\frac{1}{x} \le \frac{e^x}{x}$$

The graph $y = \frac{e^x}{x}$ lies *above* the graph $y = \frac{1}{x}$. Since $\int_0^1 \frac{1}{x}\,dx$ diverges to ∞, so must $\int_0^1 \frac{e^x}{x}\,dx$ diverge to ∞.



Comparison test

Suppose $\int_a^b f(x)\,dx$ is an improper integral with $f(x) \geq 0$, $a \leq x \leq b$.

- If you can find a function $g(x)$ such that $g(x) \geq f(x)$ for $a \leq x \leq b$, and $\int_a^b g(x)\,dx$ converges, then $\int_a^b f(x)\,dx$ must converge to some nonnegative value less or equal to $\int_a^b g(x)\,dx$.
- If you can find a function $g(x)$ such that $0 \leq g(x) \leq f(x)$ for $a \leq x \leq b$, and $\int_a^b g(x)\,dx$ diverges to ∞, then $\int_a^b f(x)\,dx$ must also diverge to ∞.

Questions

Determine if the following improper integrals converge or diverge by comparing each to an appropriate integral.

- $\int_1^\infty \frac{1}{x - e^{-x}}\,dx$

Questions

Determine if the following improper integrals converge or diverge by comparing each to an appropriate integral. If it converges, approximate it using the midpoint rule with $n = 8$.

- $\int_0^1 \frac{1}{\sqrt{x} + x}\,dx$
- $\int_0^1 \frac{1}{x^2 - x\sin(x)}\,dx$