Calculus With Parametric Equations

MAT 229, Spring 2025

Supporting materials

If you wish to get a different perspective on the notes below, try either of the following textbook sections.

- Strang's *Calculus*
  Vol II, Section 7.2: Calculus of Parametric Curves
- Stewart's *Calculus*
  Section 10.2: Calculus with parametric curves
- Boelkins/Austin/Schlicker's *Active Multivariable Calculus*
  Section 9.7: Derivatives and integrals of vector-valued functions

Review

Parametric equations are functional values of t equal to the x and y coordinates of a point in the plane:

$$x = f(t)$$
$$y = g(t)$$

(We'll soon generalize this to 3-space, adding a third coordinate -- z=h(t).)

The **parameter** in this case is *t,* and we think of this as the independent variable; we think of both x and y as the dependent variables in the problem (the coordinates of a point in the plane -- or "in space", more generally). Frequently it's useful to think of this as a motion traced out by a point (x(t),y(t)), and of t as time.

As time t varies over its domain, a point undergoes a corresponding motion in the plane.

Questions

Verify that the following parameterizations are all for the same curve.

- $x = 2\cos(3t)$, $y = 2\sin(3t)$, $0 \le t \le \pi/3$
- $x = t$, $y = \sqrt{4 - t^2}$, $-2 \le t \le 2$
- $x = 2\sin(t)$, $y = 2\cos(t)$, $-\pi/2 \le t \le \pi/2$

Tangent lines

Questions

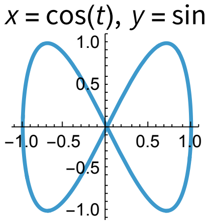
- If $y = x^2 + 1$, compute $\frac{dy}{dx}\,|_{x=1}$. What does this number represent?
- Consider the Tschirnhausen curve given by $y^2 = x^3 + 3\,x^2$.
  - Verify (1, 2) is a point on this curve.
  - Use implicit differentiation to find an equation for the tangent line to this curve at (1, 2).

Slopes of curves defined by parametric equations

Slope is rise over run. Infinitesimally, this is $\frac{dy}{dx}$. For parametric equations, the chain rule implies $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$.

Questions

- If $x = 2\,t^2 + 1,\ y = 3\,t^3 - 4$, find $\frac{dy}{dx}$.
- $x = 2\,(t - \sin(t)),\ y = 2\,(1 - \cos(t))$ are parametric equations for a cycloid. Find an equation for the tangent line to this curve where $t = \pi/6$.
- $x = \cos(t),\ y = \sin(2\,t)$ are parametric equations for the “infinity curve”:



  - Find all times for which this curve will have horizontal tangents.
  - Find all times for which this curve will have vertical tangents.
  - At what time does the curve pass through (0, 0) on this curve?

Concavity

The curve y = f(x) is concave up wherever $\frac{d^2 y}{dx^2} > 0$, and it is concave down where $\frac{d^2 y}{dx^2} < 0$. For parametric equations, we’ll use the chain rule to compute this quantity:

$$\frac{d^2 y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{dy/dt}{dx/dt}\right) = \frac{d}{dt}\left(\frac{dy/dt}{dx/dt}\right)\frac{dt}{dx}$$

Questions

- If $x = 2\,t^2 + 1$, $y = 3\,t^3 - 4$, find $\frac{d^2 y}{dx^2}$.
- x = 2 (t - sin(t)), y = 2 (1 - cos(t)) are parametric equations for a cycloid. Is the cycloid concave up or down at $t = \pi/6$?

■ $x = \cos(t)$, $y = \sin(2t)$ are parametric equations for a curve. For which values of t in $[0, 2\pi]$ is the curve concave up?

Curve length

Given parametric equations for a curve

$$x = x(t),\ y = y(t),\ a \le t \le b$$

the length of the parametric curve is

$$\int_a^b \sqrt{x'(t)^2 + y'(t)^2}\ dt.$$

Question

Find the length of the curve $x = 3\cos(t)$, $y = 3\sin(t)$, $0 \le t \le \pi$.

Note: Only on rare occasions like the above example, can these integrals be evaluated exactly. Because of the square root, usually these length integrals must often be integrated using numerical techniques.

Questions

- The cycloid $x = 2(t - \sin(t))$, $y = 2(1 - \cos(t))$ lies above the x-axis, but touches the axis occasionally. Determine the values of parameter t where it touches the x-axis.
- Find the length of one arch of the above cycloid.

Questions

Consider the parametrically defined curve

$$x = \cos(3t)$$

$$y = \sin(2t)$$

- This curve starts repeating. What is its period?
- Set up the integral to find the length of this curve.

Why the formula works

For a given curve defined by parametric equations $x = x(t)$, $y = y(t)$, $a \le t \le b$, divide $[a, b]$ into n subintervals of equal length $\Delta t = \frac{b-a}{n}$,

$$t_0 = a,\ t_1 = a + \Delta t,\ t_2 = a + 2\Delta t,\ t_3 = a + 3\Delta t,\ \ldots,\ t_n = a + n\Delta t = b$$

Approximate the length of the curve using the line segments from $(x(t_i), y(t_i))$ to $(x(t_{i+1}), y(t_{i+1}))$, as t goes from t_0 to t_n.

Question

What is the length of the line segment from point $(x(t_i), y(t_i))$ to $(x(t_{i+1}), y(t_{i+1}))$?

An approximation for the length of the curve is the sum of the lengths of these individual line segments.

$$\text{Length} \approx \sqrt{\left((x(t_0) - x(t_1))^2 + (y(t_0) - y(t_1))^2\right)} + \sqrt{\left((x(t_1) - x(t_2))^2 + (y(t_1) - y(t_2))^2\right)} + \ldots + \sqrt{\left((x(t_{n-1}) - x(t_n))^2 + (y(t_{n-1}) - y(t_n))^2\right)}$$

$$= \Sigma_{i=1}^n \sqrt{\left((x(t_{i-1}) - x(t_i))^2 + (y(t_{i-1}) - y(t_i))^2\right)}$$

The squared quantities are differences. Make them into difference quotients.

$$= \Sigma_{i=1}^{n} \sqrt{\left(\frac{(x(t_{i-1})-x(t_i))^2}{\Delta t^2}\, \Delta t^2 + \frac{(y(t_{i-1})-y(t_i))^2}{\Delta t^2}\, \Delta t^2\right)}$$

$$= \Sigma_{i=1}^{n} \sqrt{\left(\left(\frac{(x(t_{i-1})-x(t_i))^2}{\Delta t^2} + \frac{(y(t_{i-1})-y(t_i))^2}{\Delta t^2}\right) \Delta t^2\right)}$$

$$= \Sigma_{i=1}^{n} \Delta t \sqrt{\left(\left(\frac{x(t_{i-1})-x(t_i)}{\Delta t}\right)^2 + \left(\frac{y(t_{i-1})-y(t_i)}{\Delta t}\right)^2\right)}$$

For large n this becomes a better approximation. In the limit as $n \to \infty$ this expression “melts” into the integral

$$\int_a^b \sqrt{x'(t)^2 + y'(t)^2}\; dt.$$