Parametric Equations

MAT 229, Spring 2025

Supporting materials

If you wish to get a different perspective on the notes below, try either of the following textbook sections.

- Strang's *Calculus*
  Vol. II, Section 7.1: Parametric equations
- Stewart's *Calculus*
  Section 10.1: Parametric equations
- Boelkins/Austin/Schlicker's *Active Multivariable Calculus*
  Section 9.6: Vector-valued functions

Curves defined by parametric equations

One of the reasons we're interested in parametric equations is because they allow us to trace out things that aren't considered ordinary functions (they often fail the vertical line test). For example, a circle: it certainly fails the VLT!

These generalize to higher dimensions, and we can think of any orbit as a parametric curve (location as a function of time). Designers of roller coasters are all about parametric curves; automobile designers trace out beautiful curves for cars, and then engineers turn them into parametric curves that robots can trace out.

Example

The location for Tara the Tyrannosaurus Rex at time t hours is given by

$$(x, y) = (\sin(t) - \sin(2\,t), \cos(t))$$

Time t is the **parameter**, and once it is known, the dinosaur's position (x, y) is known.

Questions

- What is Tara's location at time $t = 0$?

- What is Tara's location at time $t = \pi/2$?
- When does Tara return to the location she was at when t was 0?

Definition

Parametric equations are functional values for x and y coordinates

$$x = f(t)$$
$$y = g(t)$$

usually presented as an ordered pair of coordinates in the plane:

$$(x,y) = (f(t),g(t))$$

Think of this as a curve (sometimes called a "space curve") in the plane, which is a function of t (generally thought of as time). This is an excellent way to represent a **motion**. And we can generalize this to (x,y,z) -- a curve in three-dimensional space.

Like if an apple falls on your head, or something like that -- the motion of the apple can be captured in a space curve, given by parametric equations for the motion of the apple.

Questions

Consider the parametric equations $x = \cos(t)$, $y = \sin(t)$.

- Let's make a table of values for these parametric equations for $t = 0, \pi/6, \pi/4, \pi/3, \pi/2, 2\pi/3, 3\pi/4, 5\pi/6, \pi$. Then connect these points.
- Using the Pythagorean identity, find an equation on x and y these parametric equations satisfy.
- What curve do these parametric equations represent?

Definition

A **parametric curve** is the set of all points represented by parametric equations $x = f(t)$, $y = g(t)$ for values of t over the given domain.

Graphing calculators and software like Mathematica can draw curves represented by given parametric equations.

Example

To draw the curve in Mathematica given by the parametric equations

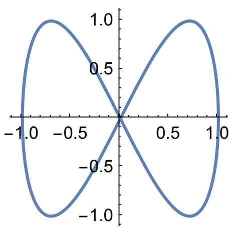
$$x = \sin(t) + \frac{1}{2}\cos(5\,t) + \frac{1}{4}\sin(13\,t)$$
$$y = \cos(t) + \frac{1}{2}\sin(5\,t) + \frac{1}{4}\cos(13\,t)$$

for $0 \le t \le 2\pi$ enter

In[635]:=
```
ParametricPlot[
 {Sin[t] + 1 / 2 Cos[5 t] + 1 / 4 Sin[13 t], Cos[t] + 1 / 2 Sin[5 t] + 1 / 4 Cos[13 t]},
 {t, 0, 2 Pi}
]
```

It’s kind of nice to watch the curve get traced out, which we can do with a “Manipulate” command in Mathematica.

Example: What’s the equation of Infinity?



Identifying curves defined by parametric equations

One may be able to identify a curve (like that infinity curve above) defined by parametric equations, by finding an equation in only x and y. Alternatively, you may want to create a motion that follows a given curve, given by x and y (e.g. you may want a robot in a bakery to trace a given curve, say laying down icing on a cake to wish someone a “Happy Birthday”).

Techniques

For the parametric equations $x = f(t)$, $y = g(t)$, two techniques for finding an equation on x and y:

- Find an identity between $f(t)$ and $g(t)$.
- Solve $x = f(t)$ -- $t=f^{-1}(x)$ -- and plug this value of t into $y = g(t) = g\left(f^{-1}(x)\right)$; or *vice versa*.

Questions

- For parametric equations $x = 2\cos(t)$, $y = 2\sin(t)$, use the Pythagorean identity, $\cos^2(t) + \sin^2(t) = 1$. What do you get?
- For parametric equations $x = 2t$, $y = 1 + t^2$, solve $x = 2t$ for t. Then relate x to y by replacing t in the equation for y. What do you get?

Questions

Identify the curves defined by the parametric equations.

- $x = 2t + 1$, $y = 3t - 4$.
- $x = \cos(t) - 5$, $y = \sin(t) + 6$.

- $x = 2\cos(t)$, $y = 3\sin(t)$.