Polar Coordinate Calculus

MAT 229, Spring 2025

- Strang's *Calculus*
  Section 7.4: Areas and Lengths in Polar Coordinates
- Stewart's *Calculus*
  Section 10.4: Areas and Lengths in Polar Coordinates

Review

Relations between Cartesian coordinates (x, y) and polar coordinates (r, θ).

$x = r\cos(\theta)$

$y = r\sin(\theta)$

and

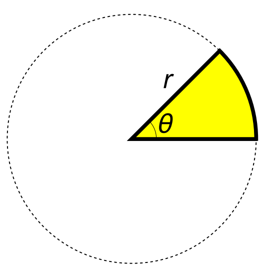
$r^2 = x^2 + y^2$

$\tan(\theta) = \frac{y}{x}$

Area

In Cartesian coordinates, area was approximated using rectangles. In polar coordinates, area is approximated using circular wedges.

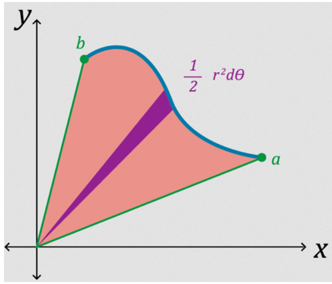
Question



What is the area of a wedge from a circle of radius r subtended by an angle θ?

Area approximation

Given a curve $r = f(\theta)$, $a \le \theta \le b$, subdivide the interval $[a, b]$ into n subintervals of equal width $\Delta\theta = \frac{b-a}{n}$ with $a = \theta_0, \theta_1, \theta_2, \ldots, \theta_{n-1}, \theta_n = b$. Approximate the region bounded by $f = f(\theta)$, $\theta_k \le \theta \le \theta_{k+1}$ with the circular wedge of radius $r = f(\theta_{k+1})$ and angle $\Delta\theta$. Sum the areas of these wedges to get an approximation for the entire region.



Area formula

The area bounded by $r = f(\theta)$, $a \le \theta \le b$ is

$$\int_a^b \frac{1}{2} r^2 \, d\theta = \int_a^b \frac{1}{2} f(\theta)^2 \, d\theta$$

Question

What is the area enclosed by the closed curve $r = 1 - \sin(\theta)$?

Questions

Consider the petaled rose $r = \cos(3\,\theta)$.

- This is a closed curve. Starting at $\theta = 0$, how far do we need to go with θ before the curve starts repeating?
- Find a range of values of θ that give one petal of the rose.
- What is the area enclosed by one petal?

Questions

We are interested in the area that is outside the circle $r = 1$ but inside circle $r = \sqrt{2}\,\sin(\theta)$.

- What are the points of intersection for these two curves?
- What range of values of θ will sweep out the area inside the second circle between these two points of intersection?
- What is the area inside the second circle but outside the first one?

Polar curves as parametric equations

Given a polar curve $r = f(\theta)$, $a \le \theta \le b$, use the equations

$$x = r\cos(\theta)$$
$$y = r\sin(\theta)$$

to get parametric equations for the curve.

Questions

Write each polar equation as parametric equations.

- $r = 2$, $0 \le \theta \le 2\,\pi$
- $r = \cos(\theta)$, $-\pi/2 \le \theta \le \pi/2$
- $r = 1 - 2\cos(\theta)$, $0 \le \theta \le 2\,\pi$

Questions

Using parametric equations for $r = \sin(\theta)$, $0 \le \theta \le \pi$, find the points in polar coordinates for the points on the curve with

- Horizontal tangents
- Vertical tangents

Questions

- For general $r = f(\theta)$ find a formula for $\frac{dy}{dx}$ in terms of r and θ.
- Use the formula to find the slope of the tangent line to $r = \cos(3\,\theta)$ at $\theta = 2\,\pi/3$.

Questions

Recall that, given parametric equations for a curve

$$x = x(t),\ y = y(t),\ a \le t \le b,$$

the length of the parametric curve is

$$\int_a^b \sqrt{x'(t)^2 + y'(t)^2}\ dt.$$

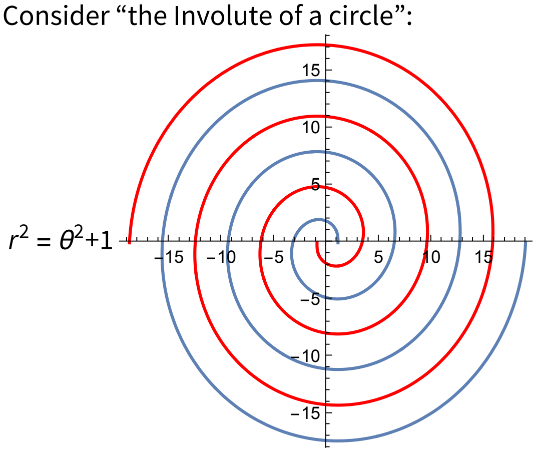
- For general $r = f(\theta)$, $a \le \theta \le b$ find a formula for its length by first writing it as parametric equations.
- Use the formula to find the length of the curve $r = \sin(\theta)$, $0 \le \theta \le \pi/2$.

Application: optimal orange peeling...

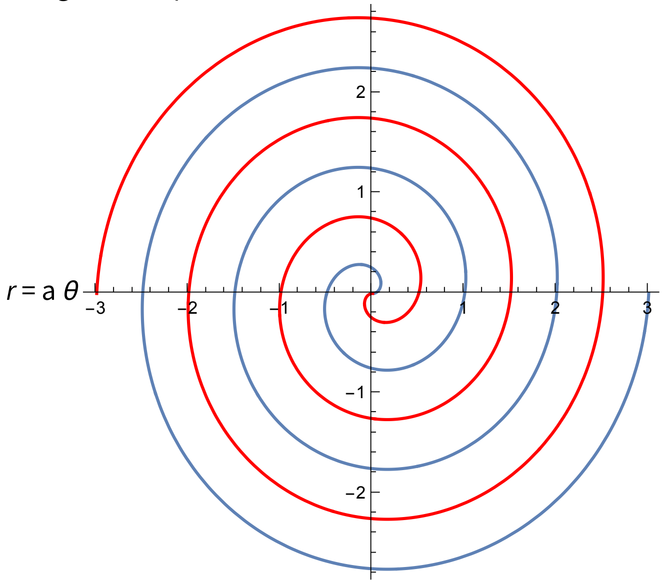
I and a few others are working on a paper about how best to peel an orange (well, it actually started out as the optimal way to create roads on a spherical planet, so that every point was within a distance D of a road, and the road was of shortest length -- but, to make it more practical, let's think of it as peeling an orange with the shortest peel and width D).

- Here is a curve that reminds me of our solution to the orange peeling problem, **if** your orange is flat and of infinite extent (aren't all of **your** oranges flat and infinite?). That reminds me of a joke about modeling elephants: "assume small spherical elephants...":)

Consider "the Involute of a circle":



- Maybe, however, my hero Archimedes may have beaten us to the solution. I'm sure that he peeled a lot of oranges. The spiral of Archimedes:



- Nonetheless I hold out hope for our peeling, which is piecewise half-circles: