# Power Series: Approximating Functions

MAT 229, Spring 2025

## Generalizing the Geometric Series

Geometric series have the form $\sum_{k=0}^{\infty} a\,r^k$. A geometric series converges if and only if $|r| < 1$. If it does converge it converges to $\frac{a}{1-r}$.
Thus, $\sum_{k=0}^{\infty} a\,r^k = \frac{a}{1-r}$, and, in particular, $\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$.

If we replace the variable "r" with the variable "x", then we obtain a **power series** for the function $\frac{1}{1-x}$. A power series is an infinite series made up of polynomial (monomial) terms, as we see here:

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k.$$

In our lab last Friday we saw that partial sums of this series worked well enough as approximations to $\frac{1}{1-x}$, so long as $|x|<1$. We call 1 the **radius of convergence** of this power series. Outside of this radius, the series diverges dramatically from the function values.

There is nothing particularly special about the function $\frac{1}{1-x}$ (although it has a very sweet power series!). In this section, we'll use it to represent other functions related to it (typically rational functions), and study the power series of a few other functions. In particular, we'll want to know under what circumstances their power series will converge (their radii of convergence). For that we often utilize the ratio test.

## Example 6.2:

### Graphing a Function and Partial Sums of its Power Series

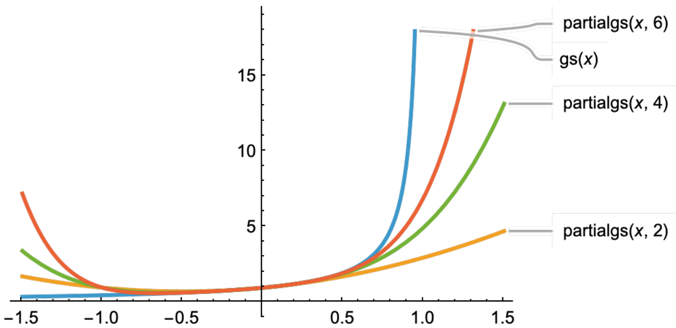
Sketch a graph of $f(x) = \frac{1}{1-x}$ and the graphs of the corresponding partial sums $S_N(x) = \sum_{n=0}^{N} x^n$ for $N = 2, 4, 6$ on the interval $(-1, 1)$. Comment on the approximation $S_N$ as $N$ increases.

It's going to be hard to force polynomials, no matter the degree, to have a vertical asymptote at x=1....

In[183]:=

```
gs[x_] := 1 / (1 - x)
partialgs[x_, n_] := Sum[x^k, {k, 0, n}]
Plot[{gs[x], partialgs[x, 2], partialgs[x, 4], partialgs[x, 6]},
 {x, -1.5, 1.5}, PlotRange → {-1, 18}, PlotLabels → Automatic]
```

Out[185]=

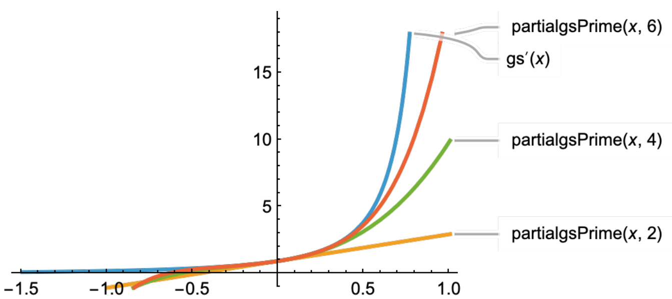



Notice that we can differentiate these series and get reasonable results, too:

In[178]:=

```
partialgsPrime[x_, n_] := Sum[k x^(k - 1), {k, 1, n}]
Plot[{gs'[x], partialgsPrime[x, 2], partialgsPrime[x, 4], partialgsPrime[x, 6]},
 {x, -1.5, 1}, PlotRange → {-1, 18}, PlotLabels → Automatic]
```

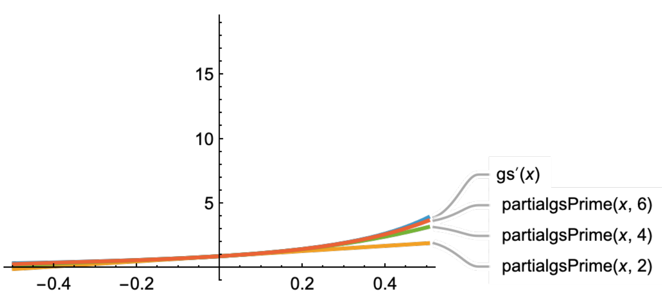
Out[179]=




In[186]:=

```
Plot[{gs'[x], partialgsPrime[x, 2], partialgsPrime[x, 4], partialgsPrime[x, 6]},
 {x, -0.5, 0.5}, PlotRange → {-1, 18}, PlotLabels → Automatic]
```

Out[186]=



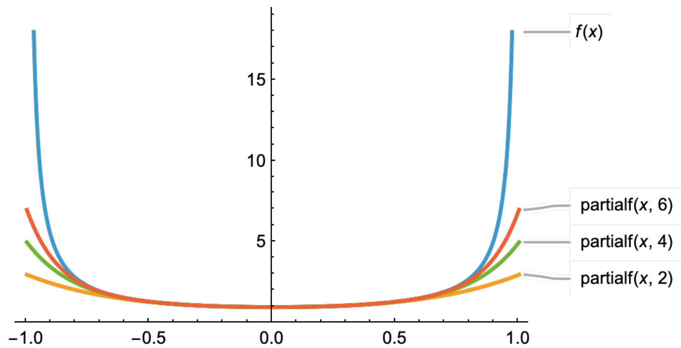

# Checkpoint 6.2:

Sketch a graph of $f(x) = \frac{1}{1-x^2}$ and the corresponding partial sums $S_N(x) = \sum_{n=0}^{N} x^{2n}$ for $N = 2, 4, 6$ on the interval $(-1, 1)$.

In[196]:=

```
f[x_] := 1 / (1 - x^2)
partialf[x_, n_] := Sum[(x^2)^k, {k, 0, n}]
Plot[{f[x], partialf[x, 2], partialf[x, 4], partialf[x, 6]},
 {x, -1, 1}, PlotRange → {0, 18}, PlotLabels → Automatic]
```

Out[198]=




# Example 6.3:

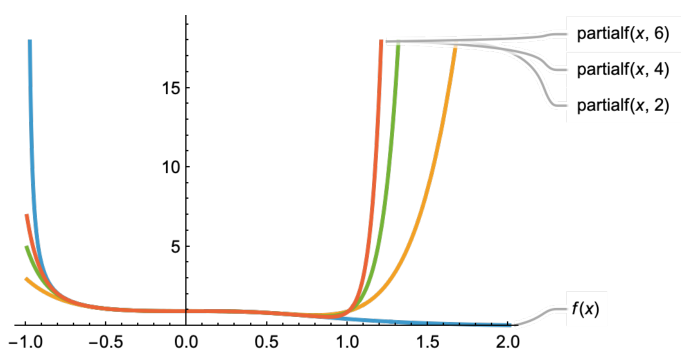
Use a power series to represent each of the following functions $f$. Find the interval of convergence.

a. $f(x) = \frac{1}{1+x^3}$
b. $f(x) = \frac{x^2}{4-x^2}$

In[202]:=

```
f[x_] := 1 / (1 + x^3)
partialf[x_, n_] := Sum[(-x^3)^k, {k, 0, n}]
Plot[{f[x], partialf[x, 2], partialf[x, 4], partialf[x, 6]},
 {x, -1, 2}, PlotRange → {0, 18}, PlotLabels → Automatic]
```

Out[204]=

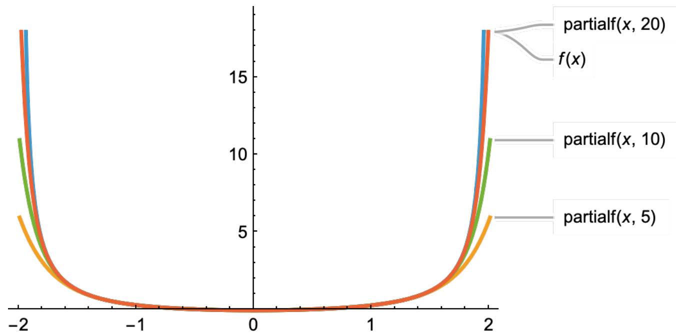



Let’s increase our partial sums a bit:

In[217]:=

```
f[x_] := x^2 / (4 - x^2)
partialf[x_, n_] := x^2 / 4 * Sum[((x / 2) ^2) ^k, {k, 0, n}]
Plot[{f[x], partialf[x, 5], partialf[x, 10], partialf[x, 20]},
 {x, -2, 2}, PlotRange → {0, 18}, PlotLabels → Automatic]
```

Out[219]=



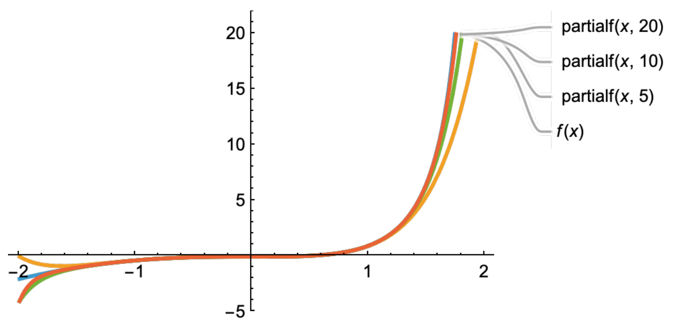

# Checkpoint 6.3:

Represent the function $f(x) = \frac{x^3}{2-x}$ using a power series and find the interval of convergence.

In[229]:=

```
f[x_] := x^3 / (2 - x)
partialf[x_, n_] := x^3 / 2 * Sum[(x / 2) ^k, {k, 0, n}]
Plot[{f[x], partialf[x, 5], partialf[x, 10], partialf[x, 20]},
 {x, -2, 2}, PlotRange → {-5, 20}, PlotLabels → Automatic]
```

Out[231]=




# Example 6.1:

For each of the following series, find the interval and radius of convergence.

a. $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

b. $\sum_{n=0}^{\infty} n! x^n$

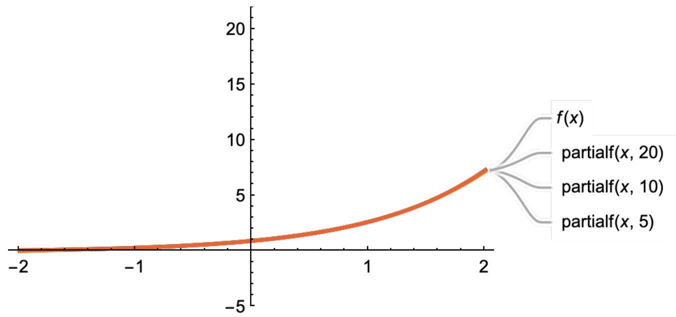
c. $\sum_{n=0}^{\infty} \frac{(x-2)^n}{(n+1)\,3^n}$

Series a: infinite radius of convergence (converges for all x), and converges to e^x:

In[232]:=

```
f[x_] := E^x
partialf[x_, n_] := Sum[x^k/k!, {k, 0, n}]
Plot[{f[x], partialf[x, 5], partialf[x, 10], partialf[x, 20]},
  {x, -2, 2}, PlotRange → {-5, 20}, PlotLabels → Automatic]
```

Out[234]=




Series c: radius of convergence (converges for all x), and converges to e^x:

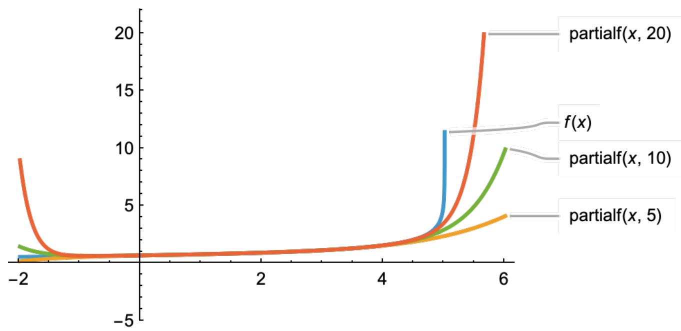
In[243]:=

```
f[x_] = Sum[((x - 2) / 3) ^k / (k + 1), {k, 0, Infinity}]
partialf[x_, n_] := Sum[(x - 2) ^k / ((k + 1) 3^k), {k, 0, n}]
partialf[x_, n_] := Sum[((x - 2) / 3) ^k / (k + 1), {k, 0, n}]
Plot[{f[x], partialf[x, 5], partialf[x, 10], partialf[x, 20]},
  {x, -2, 6}, PlotRange → {-5, 20}, PlotLabels → Automatic]
```

Out[243]=

$$\frac{\text{Log}[27] - 3\,\text{Log}[5 - x]}{-2 + x}$$

Out[246]=




Series b: radius of convergence = 0 (diverges except at x=0):

In[247]:=

```
Sum[k! x^k, {k, 0, Infinity}]
```

Out[247]=

$$\sum_{k=0}^{\infty} x^k k!$$