Power Series

MAT 229, Spring 2025

Supporting materials

If you wish to get a different perspective on the notes below, try either of the following textbook sections.

- Strang's *Calculus*
  Vol 2, Section 6.1: Power Series and functions
- Stewart's *Calculus*
  Section 11.8: Power Series
- Boelkins/Austin/Schlicker's Active Calculus
  Section 8.6: Power Series

Review

Two tests that will be the workhorses for our analysis are based on comparisons to geometric series.

Ratio test

Given any series $\sum_{k}^{\infty} b_k$, evaluate the limit of the ratio of consecutive terms, ignoring any signs, $\lim_{k\to\infty} \left| \frac{b_{k+1}}{b_k} \right|$.

- If $\lim_{k\to\infty} \left| \frac{b_{k+1}}{b_k} \right| = L$ where $L < 1$, then $\sum_{k}^{\infty} b_k$ converges absolutely.
- If $\lim_{k\to\infty} \left| \frac{b_{k+1}}{b_k} \right| = L$ where $L > 1$, then $\sum_{k}^{\infty} b_k$ diverges.
- If $\lim_{k\to\infty} \left| \frac{b_{k+1}}{b_k} \right| = L$ where $L = 1$, then you must use another convergence test.

Root test

Given any series $\sum_{k}^{\infty} b_k$, evaluate the limit of the k^{th} root, ignoring any signs, of the k^{th} term, $\lim_{k\to\infty} \sqrt[k]{|\, b_k \,|}$.

- If $\lim_{k\to\infty} \sqrt[k]{|\, b_k \,|} = L$ where $L < 1$, then $\sum_{k}^{\infty} b_k$ converges absolutely.
- If $\lim_{k\to\infty} \sqrt[k]{|\, b_k \,|} = L$ where $L > 1$, then $\sum_{k}^{\infty} b_k$ diverges.
- If $\lim_{k\to\infty} \sqrt[k]{|\, b_k \,|} = L$ where $L = 1$, then you must use another convergence test.

Power series

How can we approximate $\sin(1)$ or $e^{-1.5}$? We can represent both of these as **power series**, which are essentially series with polynomials as terms rather than just constants. So they're functions, written as infinite sums of "monomials". I hope that you're thinking "Cool!" -- but you might be thinking "Oh my God!"...:)

Turns out that a lot of our familiar functions can be written this way; furthermore, we can then approximate particular values of the functions using partial sums. In particular, many transcendental functions, like $\sin(x)$ and e^x, can be written as power series in x: **polynomials of infinite degree**.

(As you may know, nothing excites me like infinity! I get infinitely excited....)

Definition

A *power series* in x centered at a is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

where c_n is an expression in n and a is a number.

Questions

Identify the c_n and a for each of the following power series.

- $\sum_{n=2}^{\infty} (-1)^n \frac{(x-1)^n}{n}$
- $\sum_{n=1}^{\infty} \frac{(2n-1)}{n^2} x^n$
- $\sum_{n=0}^{\infty} (3x)^n$
- $\sum_{n=0}^{\infty} \frac{(5x-3)^n}{n!}$

Questions

A function is defined as the power series

$$f(x) = \sum_{n=0}^{\infty} \frac{(x-1)^n}{2^n} = 1 + \frac{x-1}{2} + \frac{(x-1)^2}{4} + \frac{(x-1)^3}{8} + \ldots$$

- What is $f(1)$?
- What is $f(2)$?
- Is $f(3)$ defined? If so, what is it? If not, why not?
- To determine the domain of this function, what do we need to worry about?

Terminology

- The *interval of convergence* is the domain for a power series.
- The *radius of convergence* is the distance from the center to the edge of the interval of convergence. It comes from the ratio/root test.

Questions

A function is defined as the power series

$$g(x) = \sum_{n=1}^{\infty} \frac{3(x+4)^n}{n}$$

- What is the center of this series?
- For what value of x is it easy to evaluate this function?

- What is the radius of convergence for this power series?
- What is the interval of convergence for this power series?

Questions

Let $H(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

- What are the first 4 terms of the series?
- What is the center of this series?
- What is the radius of convergence for this power series?
- What is the interval of convergence for this power series?
- Using an appropriate error estimate approximate $H(-1)$ with error less than 0.0001.
- What is this function? Let's plot some partial sums....
- Can you guess the function?

Questions

Let $F(x) = \sum_{n=0}^{\infty} \frac{(2x+3)^n}{n^2+1}$.

- What are the first 4 terms of the series?
- What is the center of this series?
- What is the radius of convergence for this power series?
- What is the interval of convergence for this power series?
- Using an appropriate error estimate approximate $F(-1)$ with error less than 0.0001.