

Infinite Sequences

MAT 229, Fall 2025

Supporting materials

If you wish to get a different perspective on the notes below, try either of the following textbook sections.

- Strang's Calculus:
5.1 Sequences
- Stewart's *Calculus*
11.1: Sequences
- Boelkins/Austin/Schlicker's Active Calculus
8.1: Sequences

Questions

A standard way to approximate the square root of a value is as follows. This method was known to the Babylonians, a few thousand years ago.... To approximate \sqrt{a} :

1. Make a rough estimate for the value of \sqrt{a} . Call it the first approximation x_1 .
2. The second approximation is $x_2 = 0.5\left(x_1 + \frac{a}{x_1}\right)$.
3. The third approximation is $x_3 = 0.5\left(x_2 + \frac{a}{x_2}\right)$.
4. In general, the n^{th} approximation is $x_n = 0.5\left(x_{n-1} + \frac{a}{x_{n-1}}\right)$.
 - Use this scheme to find the fourth approximation to $\sqrt{2}$ using $x_1 = 2$.
 - Do the approximations improve as n gets larger?

This method is an example of “Newton’s method”, which is a very general method for finding roots (zeros) of functions. Newton was not Babylonian however, so he must have travelled back in time.

Sequences

Definition

A *sequence* is an infinite list of terms (frequently numbers, but we’ll generalize that!) written in a definite order. We usually start counting from 1:

$$a_1, a_2, a_3, \dots, a_k, \dots$$

Examples

- The natural numbers: 1, 2, 3, 4, 5, ...
- Multiplicative inverses of the natural numbers: $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots$
- Alternating signed natural numbers: $-1, 2, -3, 4, -5, 6, -7, 8, \dots$

Not all sequences are just twists on the natural numbers, of course, but since the natural numbers serve as the indices of sequence elements, it makes sense that a lot of sequences are defined in terms of them.

Forms

Besides writing a sequence as a list we will also write some sequences

- as formulas: for example $\left\{\frac{n}{2^n}\right\}_{n=1}^{\infty} = \left\{\frac{1}{2^1}, \frac{2}{2^2}, \frac{3}{2^3}, \dots\right\}$
- recursively: for example the Fibonacci numbers: we provide two base cases, $f_1 = 1, f_2 = 1$, and then a formula for all further values ($n > 2$):
 - $f_n = f_{n-1} + f_{n-2}$
 - $\{1, 1, 2, 3, 5, 8, 13, 21, 34, \dots\}$

Limits

Frequently, but not always, sequences $\{a_n\}$ are lists of approximations that converge to some desired exact values (for example those square root approximations of the Babylonian Newton). If the limit of a_n as n tends to infinity exists,

$$\lim_{n \rightarrow \infty} a_n \text{ exists}$$

then we say that the sequence *converges*; otherwise it *diverges*.

Questions

Do any of these sequences converge? If so, to what values?

- $x_1 = 2, x_n = 0.5 \left(x_{n-1} + \frac{a}{x_{n-1}} \right)$
- $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots, \frac{1}{n}, \dots$
- $-1, 2, -3, 4, -5, 6, -7, 8, \dots, (-1)^n n, \dots$
- $1, -\frac{1}{4}, \frac{1}{9}, -\frac{1}{16}, \frac{1}{25}, -\frac{1}{36}, \dots, (-1)^{n+1} \frac{1}{n^2}, \dots$
- $\left\{\frac{n}{2^n}\right\}_{n=1}^{\infty}$
- $f_1 = 1, f_2 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for $n > 2$.

Techniques for deciding convergence

Technique 1

If $a_n = f(n)$ where f is a real-valued function, then $\lim_{n \rightarrow \infty} a_n = \lim_{x \rightarrow \infty} f(x)$.

Questions

- Find $\lim_{n \rightarrow \infty} \frac{n}{n^2+1}$ by evaluation the limit $\lim_{x \rightarrow \infty} \frac{x}{x^2+1}$.
- For which of the sequences above can you use this technique?

Technique 2

If $\lim_{n \rightarrow \infty} |a_n| = 0$, then $\lim_{n \rightarrow \infty} a_n = 0$.

Question

- Does this technique apply to any of the above limits?
- For which values of r does the sequence $\{r^n\}$ converge?

Monotonic sequences

Definition

An *increasing* sequence is one whose terms a_n satisfy $a_{n+1} \geq a_n$ for all n . It is *strictly increasing* if we can write $a_{n+1} > a_n$ for all n .

Similarly, a *decreasing* sequence is one whose terms a_n satisfy $a_{n+1} \leq a_n$ for all n . It is *strictly decreasing* if we can write $a_{n+1} < a_n$ for all n .

- What is an example of an increasing sequence?
- What is an example of a decreasing sequence?
- What is an example of a sequence that is neither increasing nor decreasing?

Definition

A sequence that is either increasing or decreasing is said to be *monotonic*.

Bounded sequences

We're familiar with bounds from our discussion of errors in numerical integration. There we established a bound on an error -- a value which was certainly larger than the error made in approximating the integral (I think of it as a "tolerance" -- how much slop you're allowed in an answer!).

- What should we mean by a sequence that is *bounded above*?
- (Hopefully you said that there's a value greater than every entry in the sequence.)
- What is an example of a sequence that is bounded above?
 - What should we mean by a sequence that is *bounded below*?
 - What is an example of a sequence that is bounded below?

Definition

A *bounded* sequence that is bounded above **and** below.

Technique 3

A bounded, monotonic sequence converges.

You might think this way: *the sequence has to go somewhere -- it's constantly increasing (say), and yet it can't go past a certain point -- so it has to go somewhere between where it is at any moment and that upper bound.* Furthermore, this is true even as long as the sequence is **eventually** monotonic.

You might think this way: *no finite chunk of a sequence at the beginning has any impact on convergence. Convergence is a property of the long tail, as the index sails off to infinity.*

The same is true for improper integrals, whose domain includes $-\infty$ or ∞ : it's all about the tail. You can rewrite one of these integrals, e.g. $\int_1^\infty \frac{1}{x^p} dx$, as $\int_1^b \frac{1}{x^p} dx + \int_b^\infty \frac{1}{x^p} dx$; the first integral is proper -- no question about its convergence -- while the second is still improper. It's the "improper part", no matter what value of $b > 1$.

Questions

Consider the sequence $\left\{\frac{n}{2^n}\right\}_{n=1}^\infty$.

- Is this sequence monotonic?
- Is this sequence bounded?