Infinite Series

MAT 229, Spring 2025

Supporting materials

If you wish to get a different perspective on the notes below, try one of the following textbook sections.

- Strang's Calculus:
  5.2 Infinite Series
- Stewart's *Calculus*
  11.2: Series
- Boelkins/Austin/Schlicker's Active Calculus
  8.1: Sequences

Review

Questions

Which of the following sequences converge? To what do they converge?

- $\left\{\frac{2n+5}{3n-1}\right\}_{n=1}^{\infty}$
- $\left\{(-1)^n \frac{3n^2+n-1}{n^2+5}\right\}_{n=0}^{\infty}$
- $\left\{\frac{6^n}{11^n}\right\}_{n=0}^{\infty}$
- $\left\{\left(-\frac{5}{8}\right)^n\right\}_{n=2}^{\infty}$
- $\left\{\left(\frac{4}{3}\right)^n\right\}_{n=1}^{\infty}$

Monotonic sequences

Definition

A sequence that is either increasing or decreasing is said to be monotonic.

Theorem:

A bounded, monotonic sequence converges.

Questions

Consider the sequence $\left\{\frac{n+1}{n}\right\}$. Write out the first few terms of this sequence.

- Is this sequence monotonic?

- Is this sequence bounded?

Questions

Consider the sequence $\left\{\frac{3^n-n}{3^n}\right\}$. Write out the first few terms of this sequence.

- Is this sequence monotonic?
- Is this sequence bounded?

Infinite sums

Many quantities can be written as infinite sums—also called **series**.

Example

In fact, any decimal number can be considered an infinite sum: consider π, for example:

$\pi = 3.14159265 \ldots = 3 + \frac{1}{10} + \frac{4}{10^2} + \frac{1}{10^3} + \frac{5}{10^4} + \frac{9}{10^5} + \frac{2}{10^6} + \frac{6}{10^7} + \frac{5}{10^8} + \ldots$

Questions

- What is the decimal version of $\frac{1}{3}$?
- Write $\frac{1}{3}$ as an infinite sum.
- Write $\frac{1}{3}$ as an infinite sum using summation notation.
- As you go farther out in the sum what is happening to the individual terms?

Questions

- What is the value of the infinite sum $\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ldots$?
- Write this infinite sum using summation notation.
- As you go farther out in the sum what is happening to the individual terms?

Partial sums

The **sequence of partial sums** for the series $a_1 + a_2 + a_3 + a_4 + \ldots$ are

$$S_1 = a_1 = \sum_{k=1}^{1} a_k$$

$$S_2 = a_1 + a_2 = \sum_{k=1}^{2} a_k$$

$$S_3 = a_1 + a_2 + a_3 = \sum_{k=1}^{3} a_k$$

$$S_4 = a_1 + a_2 + a_3 + a_4 = \sum_{k=1}^{4} a_k$$

$$\vdots$$

Questions

- What are the first 4 partial sums for the infinite series for
$$\pi = 3 + \frac{1}{10} + \frac{4}{10^2} + \frac{1}{10^3} + \frac{5}{10^4} + \frac{9}{10^5} + \frac{2}{10^6} + \frac{6}{10^7} + \frac{5}{10^8} + \ldots?$$
- What do these partial sums represent?

Definition

The infinite sum $\sum_{k=1}^{\infty} a_k$ *converges* if and only if its sequence of partial sums converge. If it converges, its value is the limit of the partial sums:

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=1}^{n} a_k.$$

(Note the similarity to improper integrals of the form $\int_a^\infty f(x)\,dx = \lim_{R\to\infty} \int_a^R f(x)\,dx$.)

Question

Consider the series $\sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1}\right)$.

- What are the first four partial sums for this series?
- What is the value of the n^{th} partial sum?
- Does this series converge? If so, to what value?

Geometric series

Definition

A geometric series has the form $\sum_{k=n_0}^{\infty} a\,r^k$ for some numbers n_0, a, and r.

Questions

Which of the following are geometric series?

- $\sum_{k=0}^{\infty} 2\left(\frac{1}{3}\right)^k$
- $\frac{7}{2} + \frac{7}{4} + \frac{7}{8} + \frac{7}{16} + \frac{7}{32} + \frac{7}{64} + \ldots$
- $\sum_{k=}^{\infty} (3^k \times 2^{-k})$

Convergence/divergence

Consider the geometric series $\sum_{k=0}^{\infty} r^k$. Let $S_n = \sum_{k=0}^{n-1} r^k = 1 + r + r^2 + \ldots + r^{n-1}$ be the n^{th} partial sum for this series.

- The n^{th} partial sum times r is $r\,S_n = r \sum_{k=0}^{n-1} r^k = r\left(1 + r + r^2 + \ldots + r^{n-1}\right) = r + r^2 + \ldots + r^n$
- In the difference $S_n - r\,S_n$ most terms cancel: **all** but the first term in S_n and the last term in $r\,S_n$.
$$S_n - r\,S_n = \left(1 + r + r^2 + \ldots + r^{n-1}\right) - \left(r + r^2 + \ldots + r^n\right) = 1 - r^n$$

- This is an equation we can solve for the unknown S_n.

$$S_n - r\, S_n = 1 - r^n$$
$$\longrightarrow S_n(1-r) = 1 - r^n$$
$$\longrightarrow S_n = \frac{1-r^n}{1-r}, \text{ if } r \neq 1$$

- The infinite sum's convergence or divergence is equivalent to the convergence or divergence of $\frac{1-r^n}{1-r}$ as $n \to \infty$. In the "Sequences" handout, this was a question about $\lim_{n\to\infty} r^n$. There we computed

$$\lim_{n\to\infty} r^n \begin{cases} \text{converges to } 0 & \text{if } |r| < 1 \\ \text{converges to } 1 & \text{if } r = 1 \\ \text{diverges otherwise} \end{cases}$$

  - If $|r| < 1$, then $\sum_{k=0}^{\infty} r^k = \lim_{n\to\infty} \frac{1-r^n}{1-r} = \frac{1}{1-r}$
  - If $r = 1$, the partial sum formula doesn't make sense since it has a zero divide. The partial sum in this case is

    $$S_n = 1 + (1) + (1)^2 + (1)^3 + \ldots + (1)^{n-1} = n$$

    This means $\sum_{k=0}^{\infty} (1)^k = \lim_{n\to\infty} n = \infty$
  - The geometric sum diverges for all other values of r.

Questions

Which of the geometric series converge? For those which do converge, to what value do they converge?

- $\sum_{k=0}^{\infty} 2\left(\frac{1}{3}\right)^k$
- $\frac{7}{2} + \frac{7}{4} + \frac{7}{8} + \frac{7}{16} + \frac{7}{32} + \frac{7}{64} + \ldots$
- $\sum_{k=1}^{\infty} 2^{-k} \times 3^k$

Questions

- Consider the geometric sum $\sum_{n=0}^{\infty} 4\left(\frac{5}{6}\right)^n$
  - What are the first few terms of this sum?
  - What is the value of this infinite sum?
- Consider the geometric sum $\sum_{n=2}^{\infty} 3\left(-\frac{3}{5}\right)^n$
  - What are the first few terms of this sum?
  - Rewrite this sum in the form $\sum_{n=0}^{\infty} a\, r^n$. What is a? What is r?
  - What is the value of this infinite sum?
- Consider the sum $\sum_{n=0}^{\infty} \frac{2^n+5^n}{7^n}$.
  - What are the first few terms of this sum?
  - Rewrite this infinite sum as the sum of two geometric series
  - What is the value of this infinite sum?

Repeating decimals

Any decimal number that has a repeating pattern can be written as a fraction. For example, the decimal 5.121212121212… has the 12 repeating behavior. We can represent these numbers as geometric series and find the sums' values as fractions.

$$5.121212121212\ldots = 5 + \frac{12}{100} + \frac{12}{100^2} + \frac{12}{100^3} + \ldots$$
$$= 5 + \Sigma_{n=0}^{\infty} \frac{12}{100} \left(\frac{1}{100}\right)^n = 5 + \frac{12}{100} \frac{1}{1-1/100}$$
$$= 5 + \frac{12}{100} \frac{1}{99/100} = 5 + \frac{12}{99} = 5 + \frac{4}{33} = \frac{169}{33}$$

Question

- What is a fraction equal to 0.7777777 ...?
- What is a fraction equal to 34.123123123123123123 ...?

Divergence Theorem

Questions

Which of the following sums converge? What is true about what happens to the individual terms of the sums?

- $\Sigma_{n=1}^{\infty} 2 = 2 + 2 + 2 + 2 + \ldots$
- $\Sigma_{k=0}^{\infty} \left(\frac{1}{2}\right)^k = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$
- $\Sigma_{k=1}^{\infty} \frac{k}{k+1} = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \ldots$

Questions

Suppose the series $a_1 + a_2 + a_3 + a_4 + \ldots$. converges to L.

- What is $S_{n+1} - S_n$?
- What is $\lim_{n\to\infty} S_{n+1}$? What is $\lim_{n\to\infty} S_n$? What is $\lim_{n\to\infty} S_{n+1} - S_n$?

Theorem

If $\lim_{k\to\infty} a_k \neq 0$, then the infinite sum $\sum_{k}^{\infty} a_k$ cannot converge.

Question

How do I know that $\sum_{k=1}^{\infty} \frac{k}{2k+1}$ diverges?

Major note

The divergence test *only* gives conclusive information if the limit of the individual terms does NOT go to 0. If the individual terms go to 0, anything is possible.

Examples

- $\Sigma_{k=0}^{\infty}\left(\frac{1}{3}\right)^k = 1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \ldots$
  Here the individual terms are $\frac{1}{3^k}$, and we know $\lim_{k\to\infty} \frac{1}{3^k} = 0$. Also, we can recognize this as a geometric series with $r = \frac{1}{3} < 1$. It must converge.

- $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} + \ldots$
  Here the individual terms are $\frac{1}{\sqrt{k}}$, and we know $\lim_{k\to\infty} \frac{1}{\sqrt{k}} = 0$. Let's compute a few partial sums that use progressively more and more terms. From them it appears the partial sums are getting arbitrarily large so that the series diverges to infinity. We will verify this pattern is true later.
  - S_{10}
  - S_{20}
  - S_{30}
  - S_{100}
  - S_{1000}
  - $S_{10\,000}$