Section Summary: Calculus with Parametric Curves

a. Definitions

None to speak of.

b. Theorems

If a curve C is described by the parametric equations $x = f(t)$, $y = g(t)$, $\alpha \leq t \leq \beta$, where f' and g' are continuous on $[\alpha, \beta]$ and C is traversed exactly once as t increases from α to β, then the length of C is

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}\, dt$$

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$$S = \int_{\alpha}^{\beta} 2\pi y \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}\, dt$$

c. Properties/Tricks/Hints/Etc.

It is important to distinguish between the derivatives with respect to the parameter t and with respect to x: in general

$$\frac{dy}{dt} \neq \frac{dy}{dx}$$

One reflects the time rate of change in y; the other indicates the tangential rate of change in the curve of $y = f(x)$.

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d. Summary

Suppose that we can write our parametric curves as $y = F(x)$:

$$\begin{matrix} x = f(t) & & \\ & \longrightarrow & y = F(x) \\ y = g(t) & & \end{matrix}$$

For example,

$$\begin{matrix} x = sin(t) & & \\ & \longrightarrow & y = x^2 \\ y = sin^2(t) & & \end{matrix}$$

Then, using the chain rule,

$$F'(x) = F'(f(t)) = \frac{g'(t)}{f'(t)}$$

or

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

Problem #67, p. 677 shows that this is often possible: "If f' is continuous and $f'(t) \neq 0$ for $a \le t \le b$, show that the parametric curve $x = f(t)$, $y = g(t)$, $a \le t \le b$ can be put in the form $y = F(x)$."

f' must be of fixed sign (positive or negative) on $[a, b]$, as it is continuous and never zero (by the intermediate value theorem).

Thus f is strictly monotonic (either increasing or decreasing) on the interval, and hence one-to-one $[f(x_1) = f(x_2) \Longrightarrow x_1 = x_2]$ on $[a, b]$.

Therefore the inverse f^{-1} exists and is continuous on $[f(a), f(b)]$, so

$$t = f^{-1}(x)$$

and

$$y = g(t) = g(f^{-1}(x)) \equiv F(x).$$

That is,

$$F = g \circ f^{-1}$$

So what's the problem with

$$\begin{matrix} x = sin(t) \\ y = sin^2(t) \end{matrix} \quad \longrightarrow \quad y = x^2$$

The problem is that the same curve is used over and over as the path for the parametric motion. The point turns around whenever $x'(t) = \cos(t) = 0$: at those times, the motion is stopped, and the point is turning around on the parabola.

Much of this is just about change of variable. We start with our old formulas: for example, if f' is continuous on $[a, b]$, then the length of the curve $y = f(x)$, $a \leq x \leq b$, is

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

Now, suppose that x and y are given parametrically, as $x = f(t)$ and $y = g(t)$. Then

$$L = \int_\alpha^\beta \sqrt{1 + \left(\frac{dy/dt}{dx/dt}\right)^2} \frac{dx}{dt} dt$$

where $\alpha = f^{-1}(a)$ and $\beta = f^{-1}(b)$. (Note: since f^{-1} must exist, x is travelling from left to right or from right to left, and doesn't stop; that is, $\frac{dx}{dt} \neq 0$).