**Section Summary: 11.9 – Functions as Power Series**

# 1 Definitions

# 2 Theorems

**term by term differentiation and integration** If the power series

$$\sum c_n(x-a)^n$$

has radius of convergence $R > 0$, then the function $f$ defined by

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots = \sum_{n=1}^{\infty} c_n(x-a)^n$$

is differentiable (and therefore continuous) on the interval $(a-R, a+R)$ and

a.

$$f'(x) = c_1 + 2c_2(x-a) + \ldots = \sum_{n=0}^{\infty} nc_n(x-a)^{n-1}$$

b.

$$\int f(x)dx = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + \ldots = C + \sum_{n=0}^{\infty} c_n\frac{(x-a)^{n+1}}{n+1}$$

The radii of convergence of the integrated and differentiated power series are both $R$.

We do have to check the endpoints of the interval of convergence however: differentiation tends to roughen things up, so if an endpoint is convergent for $f$, it may not be for $f'$; whereas integration tends to smooth things out, so if an endpoint is divergent for $f$, it may be "healed" by the anti-derivative of $f$.

# 3 Properties, Hints, etc.

Notice the "Leibniz formula" for $\pi$ (p. 774): since it's an alternating series, we find $\pi$ to any accuracy we desire, by simply making the first neglected term small enough.

## 4 Summary

We can construct new power series from old ones in several ways: by

a. writing them as composite functions,

b. integration

c. differentiation

The radius of convergence in the last two cases stays the same as the original power series; in the case of composition, we need to recalculate the radius of convergence.

The interval of convergence does not necessarily remain the same, however: in particular, you still have to check the ends separately, as noted above.

Notice how power series can be used to integrate (approximately) complicated function, and then provide a mechanism for determining the error in the approximation.